On the Center of Mass of the Half *n*-Ball

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ABSTRACT
This project explores how the center of mass (COM) of a half n-ball depends on the dimension, n. We study the case of uniform density, where the COM is equivalent to the object's centroid, or geometric center. We find a closed form solution, a sequence describing the centroid in n dimensions, which confirms the common two and three dimensional cases. Furthermore, the sequence is analytically proven to converge to zero in the limit as n approaches infinity.

KEYWORDS
Mechanics; Mathematical Physics; Center of Mass; Centroid; Hyper-Geometry

INTRODUCTION

When first learning mechanics we often begin by studying the motion of a point object. In general, determining the motion of an extended object is more involved since they are essentially the collection of many individual point masses. The way we can simplify these calculations is by treating the object as if all its mass were at a single point, its center of mass (COM). As one would expect, the COM of the object depends not only on the shape of the object, but also the mass distribution. For example, the COM of a ball with uniform density will be different than the COM of a ball that has one hemisphere with a higher density than the other.

Since we can learn so much about the motion of a system by studying its COM, it is a common tool in many fields of physics. For example, in many cases when studying orbital dynamics it is sufficient to consider the motion of the centers of mass of the orbiting bodies. We find the COM used in many engineering applications as well. Specifically, it is a great tool in designing buildings to be stable to seismic perturbation. By designing counterweights and balancing systems, it is possible to create systems which restore a building's position given a perturbation to its foundation.[1] A particularly interesting application of the COM comes from the field of biomechanics. One of the most common models of balance is called the 'inverted pendulum model' in which the leg is approximated as a point mass situated above the foot and connected by a rod.[2,3] Moreover, the COM trajectory (in the sagittal plane) is of extreme importance when discussing the equilibrium of a person with prosthetics or humanoid robots, and such complex calculations are performed using neural networks.[4] Furthermore, the calculations of the COM in n-dimensions is a good introduction for students interested in string theory or in mathematics.[5]

Mathematically, the COM, C of an arbitrary massive object O is given by

$$C = \frac{1}{M}\int_O \vec{r}\,dm,$$

where M is the total mass of the object, and $\vec{r}$ is the vector pointing from the origin to the mass element dm.[6] Notice that this is simply the mass-averaged position vector. For our purposes we consider only the case of a uniform density

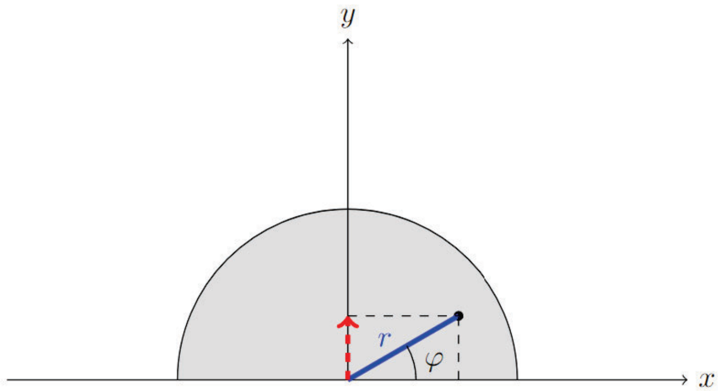

Figure 1. Figure representing the two dimensional case. The length, r, and the angle, φ, show an arbitrary position vector lying in the disk. The dashed, red vector represents the y-component which will be used for calculations.

distribution, in which case we can rewrite this integral in terms of volume by substituting $M = \rho V$, $dm = \rho \, dV$, where ρ is the density and V is the total volume of the object. Then the integral simplifies to the volume averaged position vector,

$$C = \frac{1}{V} \int_V \vec{r} \, dV \qquad \textbf{Equation 1.}$$

Therefore, if the object has uniform density, then the COM is entirely equivalent to the geometric center of the object, or, its *centroid*.

Consider the following two examples, in which the COM of a half ball is found for two familiar cases: two and three dimensions (adapted from problems 3.21 and 3.22 from Taylor).[6]

Two Dimensional Case

In the 2D case we have a half disk of radius R and a uniform mass distribution. The base of the disk is situated along the x-axis with its center at the origin. A visualization of this system is shown as **Figure 1**. Due to the geometry of the disk, it is more appropriate to use polar coordinates, where,

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$dA = r \; dr d\varphi.$$

From the symmetry of the object we know that the COM must lie along the axis of symmetry, the y-axis. This allows us to simply replace $\vec{r}$ (**Equation 1**) with the y-coordinate, $y = r\sin\varphi$. Thus,

$$C = \frac{1}{A}\int y \; dA = \frac{2}{\pi R^2}\int_0^\pi \int_0^R r^2 \sin\varphi \; dr d\varphi = \frac{4}{3\pi}R \approx 0.424R.$$

Notice that C represents only the y-component to the COM, however by our choice of coordinates the x-component is 0 so C sufficiently defines the entire COM vector.

Three Dimensional Case

In this case there is a half sphere of radius R and a uniform mass distribution. The base of the half sphere lies on the xy-plane with its center at the origin. This system is represented in **Figure 2**. Of course, it is best to use spherical coor-

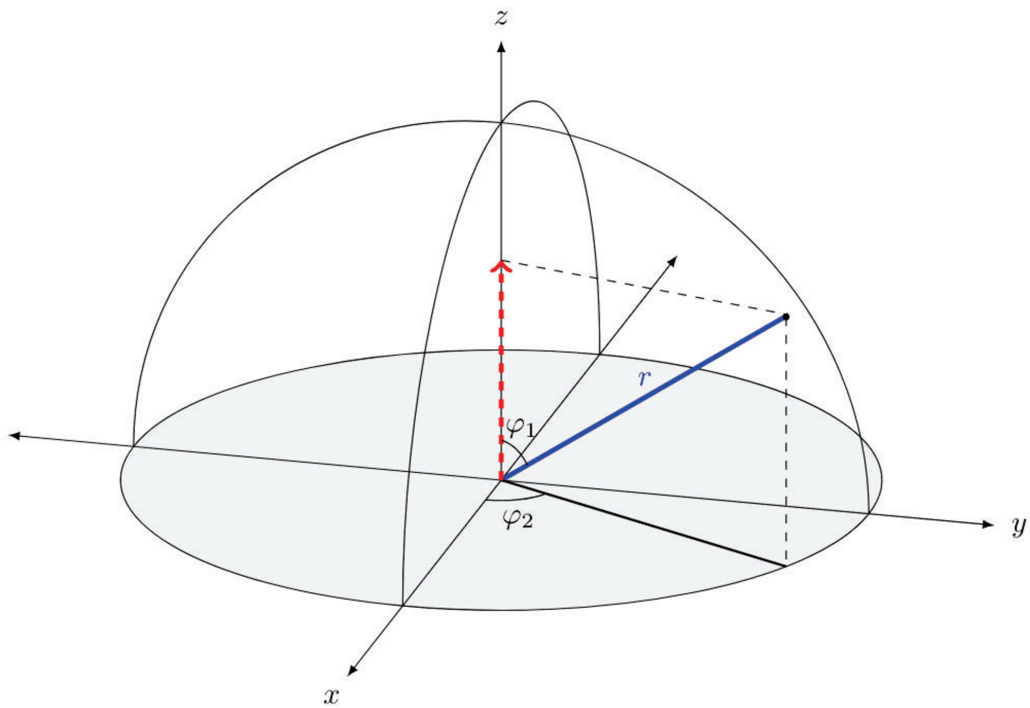

Figure 2. This figure represents 3-dimensional spherical coordinates, where φ_1 is measured from the positive z-axis and φ_2 is measured from the positive x-axis. Similarly to Figure 1, the dashed, red vector represents the component which we use to calculate the COM, in this case, the z-component.

dinates, where,

$$\begin{aligned} x &= r \sin\varphi_1 \cos\varphi_2 \\ y &= r \sin\varphi_1 \sin\varphi_2 \\ z &= r \cos\varphi_1 \\ dV &= r^2 \sin\varphi_1 \; dr d\varphi_1 d\varphi_2. \end{aligned}$$

Similarly to the 2D case, symmetry dictates that the COM lies along the z-axis. Then,

$$C = \frac{1}{V}\int z \; dV = \frac{3}{2\pi R^3}\int_0^{\pi/2}\int_0^{2\pi}\int_0^R r^3 \cos\varphi_1 \sin\varphi_1 \; dr d\varphi_2 d\varphi_1 = \frac{3}{8}R = 0.375R.$$

Notice that similarly to the two dimensional case, the COM is a vector in 3-dimensional space, but since the z-component is the only non-zero component, C sufficiently describes the COM. This convention will be used for the general n-dimensional case as well.

If we compare the two and three dimensional cases we find that $C_2 = 0.424 > 0.375 = C_3$, or in words, when we moved up one dimension the COM became closer to the origin. This begs the question of what happens when we continue increasing the dimension. In this paper we look to answer the following two questions:

1. As the number of dimensions increases, does the COM continue moving closer to the origin?

2. If so, does the COM of the half n-ball converge to zero as intuition would predict?

In principle, we could examine the trend in higher dimensions by directly computing the fourth, fifth, and even higher dimensional cases individually. However, as we will see, we can analytically solve the problem without ever specifying the dimension n, and as in any problem in physics, generalizing the solution leads to deeper understanding. This theoretical exercise will not only yield the general solution, but it will give us insight into the pattern we see at lower dimensions. In the following sections we will address the above questions by constructing a closed form describing the COM in the arbitrary n-dimensional case. Since the domain of this function is the natural numbers, this solution is a sequence and so we may then use techniques from calculus and analysis to study the convergence of this sequence.

RESULTS

The n Dimensional Case

When we generalize the above examples to n-dimensions, we must work in n-dimensional spherical coordinates. The generalization is fairly obvious with Cartesian coordinates. Let $\{e_i \mid i \in I\}$ be a set of orthonormal vectors, where I is our indexing set. Then in n-dimensional space a vector $\vec{x}$ can be written as $\vec{x} = \xi_1\ e_1 + \xi_2\ e_2 + \ldots + \xi_n\ e_n$, where ξ_i is the component of $\vec{x}$ in the e_i direction[7]. In spherical coordinates however, the vector $\vec{x}$ can be expressed in terms of one radial component, r, and $n-1$ angular coordinates as $\vec{x} = (r, \varphi_1, \varphi_2, \ldots, \varphi_{n-1})$, where $r \geq 0$, $\varphi_j \in [0, \pi]$ for $j \in \{1, 2, \ldots, n-2\}$, and $\varphi_{n-1} \in [0, 2\pi)$. Just as in the lower dimensional cases, trigonometry can be used to change bases and convert from spherical to Cartesian coordinates. The conversion is given by **Equations 2, 3, 4, and 5** below,[7]

$$\xi_1 = r\cos\varphi_1 \qquad \textbf{Equation 2.}$$

$$\xi_j = r\cos\varphi_j \prod_{k=1}^{j-1} \sin\varphi_k \qquad (j = 2, \ldots, n-2) \qquad \textbf{Equation 3.}$$

$$\xi_{n-1} = r\sin\varphi_{n-1} \prod_{k=1}^{n-2} \sin\varphi_k \qquad \textbf{Equation 4.}$$

$$\xi_n = r\cos\varphi_{n-1} \prod_{k=1}^{n-2} \sin\varphi_k \qquad \textbf{Equation 5.}$$

It may not be immediately clear where these change-of-basis equations come from, however understanding it is critical to determining the centroid of the half hyper-sphere. In two and three dimensions the axis of symmetry was obvious visually, however visual arguments no longer suffice when working in higher dimensional spaces. That being said, the three dimensional case can help give us *some* visual intuition. It is clear from the **Equations 2-5** above that φ_1 is the angle ranging from 0 to π, measured from the positive e_1 axis. Due to this, the $\cos\varphi_1$ term of ξ_1 shows that it is the projection of $\vec{x}$ onto the e_1 axis. Similarly, φ_2 is orthogonal to φ_1 and is measured from the positive e_2 axis. This means that written out, $\xi_2 = r\sin\varphi_1\cos\varphi_2$. In this case, the $\sin\varphi_1$ term projects $\vec{x}$ onto the $n-1$ dimensional subspace orthogonal to e_1, and the $\cos\varphi_2$ term then projects it onto the e_2 axis. This process continues so that the ξ_j component is projection of $\vec{x}$ onto the $n-(j-1)$ dimensional subspace spanned by $\{e_j, e_{j+1}, \ldots, e_{n-1}, e_n\}$, and then projected onto the e_j axis. This ends when we are left only with a two dimensional subspace spanned by $\{e_{n-1}, e_n\}$, with φ_{n-1} running from 0 to 2π measured from the positive e_n to the positive e_{n-1} axis. This explains why the only difference between ξ_{n-1} and ξ_n is the $\cos\varphi_{n-1}$ term on ξ_n and the $\sin\varphi_{n-1}$ term on ξ_{n-1}. For a concrete example, consider $n = 3$. Then, $(\xi_1, \xi_2, \xi_3) = (z, y, x)$, φ_2 is the angle running from the positive x-axis to the positive y-axis, and φ_1 is the angle from the positive z-axis to the xy-plane.

In order to calculate the centroid according to **Equation 1** above, we need the volume of an n-dimensional half ball,

V_n, and the n-dimensional spherical volume element, dV. From Blumenson,[7]

$$dV = r^{n-1} \prod_{k=1}^{n-2} \sin^k(\varphi_{n-1-k})\ dr d\varphi_1 d\varphi_2 \dots d\varphi_{n-1}$$
$$dV = r^{n-1}\ dr\ \ \sin^{n-2}(\varphi_1)\sin^{n-3}(\varphi_2)\dots\sin(\varphi_{n-2})\ d\varphi_1 d\varphi_2 \dots d\varphi_{n-1} \quad \text{Equation 6.}$$

Additionally, from Smith[8] and Wang[9] we have that the n-ball of radius R defined by the set

$$B_n = \{\mathbf{x} = (x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 \le R^2\}$$

has the volume

$$V(B_n) = \frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})} R^n, \quad \text{Equation 7.}$$

In the formula, $\Gamma(n/2)$ represents Euler's gamma function evaluated at $n/2$.

We can simplify **Equation 7** using some gamma function identities. The gamma function is Euler's extension of the factorial function to non integer arguments. As such, it satisfies some properties that match the behavior of factorial function. For example,[10] for any $x > 0$,

$$x\Gamma(x) = \Gamma(x+1), \quad \text{Equation 8.}$$

and in particular, if n is a natural number then,

$$\Gamma(n+1) = n! \quad \text{Equation 9.}$$

just as we would expect. The two properties shown by **Equations 8** and **9** above can be used to show that for the case of even n,

$$\Gamma\left(\frac{n}{2}\right) = \left(\frac{n}{2} - 1\right)! \quad \text{Equation 10.}$$

In the case that n is odd,

$$\Gamma\left(\frac{n}{2}\right) = \frac{\pi^{\frac{1}{2}}(n-2)!!}{2^{\frac{(n-1)}{2}}} \quad \text{Equation 11.}$$

Equation 10 is a direct result of **Equation 9**, and **Equation 11** can be proven easily via induction. Now that we have closed form solutions for the gamma function evaluated at integer and half integer arguments, we can plug into **Equation 7** and divide by two to get the volume of the *half* n ball. Since $\Gamma(\frac{n}{2})$ depends on the parity of n, then the volume of the half n ball, which we denote by V_n, is given by the following piecewise formula,

$$V_n = \begin{cases} \frac{\pi^{n/2}}{n(\frac{n}{2}-1)!} R^n & \text{for even n} \\ \\ \frac{(2\pi)^{[(n-1)/2]}}{(n)!!} R^n & \text{for odd n} \end{cases} \quad \text{Equation 12.}$$

Now that we are equipped with the adequate tools, we are finally prepared to solve **Equation 1** in the arbitrary n-dimensional case. Recall **Equation 1**

$$C = \frac{1}{V}\int_V \vec{r}\ dV.$$

For the half n-ball, this integral turns into,

$$C = \frac{1}{V_n}\int_{V_n} \vec{r}\ r^{n-1} \prod_{k=0}^{n-2} \sin^k(\varphi_{n-1-k})\ d\varphi_{n-1-k}. \quad \text{Equation 13.}$$

Recall from the two and three dimensional cases that if a convenient choice of axes is chosen, we can force the axis of symmetry to lie along one of the Cartesian unit vectors. In other words, if we align the axis of symmetry with one of the bases of the Cartesian coordinate system, say e_k, then $\vec{r}$ can be replaced with ξ_k. Recall that for n-dimensional spherical coordinates, angular components φ_1 through φ_{n-2} run from 0 to π, and φ_{n-1} runs from 0 to 2π. To create a half ball we simply restrict the domain of one of the angles. For the sake of simplicity, let φ_1 run only from 0 to $\pi/2$, thus making ξ_1 the component which lies along the axis of symmetry. This turns **Equation 13** into,

$$\begin{aligned} C &= \frac{1}{V_n}\int_{V_n} r\cos(\varphi_1)\; r^{n-1}\prod_{k=0}^{n-2}\sin^k(\varphi_{n-1-k})\; d\varphi_{n-1-k} \\ &= \frac{1}{V_n}\int_0^R r^n\; dr\int_0^{\pi/2}\cos(\varphi_1)\sin^{n-2}(\varphi_1)\; d\varphi_1 \\ &\quad \int_0^\pi \dots \int_0^\pi \prod_{k=1}^{n-3}\sin^k(\varphi_{n-1-k})\; d\varphi_{n-1-k}\int_0^{2\pi} d\varphi_{n-1}, \end{aligned}$$ **Equation 14.**

For $n > 1$, integrating by parts gives us that

$$\int_0^{\pi/2}\cos(\varphi_1)\sin^{n-2}(\varphi_1)\; d\varphi_1 = \frac{1}{n-1}$$ **Equation 15.**

By using **Equation 15** and integrating the radial and φ_{n-1} terms, we simplify **Equation 14** to the following,

$$C = \frac{2\pi R^{n+1}}{(n+1)V_n}\frac{1}{n-1}\int_0^\pi \dots \int_0^\pi \prod_{k=1}^{n-3}\sin^k(\varphi_{n-1-k})\; d\varphi_{n-1-k}$$ **Equation 16.**

Now all that remains is to solve the multiple integral at the end of **Equation 14**. Notice that for any k, $\sin^k(\varphi_{n-1-k})$ is a function of only one variable, and so we may use Fubini's theorem to split this multiple-integral of a product into a product of single integrals, or,

$$\int_0^\pi \dots \int_0^\pi \prod_{k=1}^{n-3}\sin^k(\varphi_{n-1-k})\; d\varphi_{n-1-k} = \prod_{k=1}^{n-3}\int_0^\pi \sin^k(\varphi_{n-1-k})\; d\varphi_{n-1-k}$$ **Equation 17.**

Now, using the power reduction formula we find,

$$\begin{aligned} \int_0^\pi \sin^n(x)\; dx &= -\frac{1}{n}\sin^{n-1}(x)\cos(x)\Big|_0^\pi + \frac{n-1}{n}\int_0^\pi \sin^{n-2}(x)\; dx \\ &= \frac{n-1}{n}\int_0^\pi \sin^{n-2}(x)\; dx. \end{aligned}$$ **Equation 18.**

Thus, as a result of our bounds, we have a recursive relation describing the solution to the integral. This recursion relation can be repeatedly applied to attain closed form solutions to **Equation 18**. If n is even then,

$$\int_0^\pi \sin^n(x)\; dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\dots\left(\frac{3}{4}\right)\int_0^\pi \sin^2(x)\; dx = \frac{(n-1)!!}{n!!}\pi,$$ **Equation 19.**

and if n is odd then,

$$\int_0^\pi \sin^n(x)\; dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\dots\left(\frac{2}{3}\right)\int_0^\pi \sin(x)\; dx = \frac{(n-1)!!}{n!!}2.$$ **Equation 20.**

More concisely, we have,

$$\int_0^\pi \sin^n(x)\; dx = \begin{cases} \frac{\pi(n-1)!!}{n!!} & \text{for even } n \\ \\ \frac{2(n-1)!!}{n!!} & \text{for odd } n \end{cases}$$ **Equation 21.**

We can now use the above equation to simplify **Equation 17**. From **Equation 21** we see that the only difference between the odd and the even case is whether the integral carries a factor of 2 or a factor of π. Consequently, when we take the product of these integrals it is important to count how many odd terms and how many even terms are in each product. Notice that if n is even, then $n-3$ is odd, which means that the product has $\frac{n-4}{2}+1$ terms with k odd, and $\frac{n-4}{2}$ terms with k even. Similarly, in the case that n is odd, then $n-3$ is even, meaning there are $\frac{n-3}{2}$ terms with k even and $\frac{n-3}{2}$ terms with k odd. Therefore, putting this all together we have that,

$$\prod_{k=1}^{n-3}\int_0^{\pi}\sin^n(x)\ dx = \begin{cases} 2^{(n-4)/2+1}\pi^{(n-4)/2}\left[\frac{(n-4)!!}{(n-3)!!}\frac{(n-5)!!}{(n-4)!!}\cdots\frac{(2)!!}{(3)!!}\frac{(0)!!}{(1)!!}\right] & \text{for even n} \\ 2^{(n-3)/2}\pi^{(n-3)/2}\left[\frac{(n-4)!!}{(n-3)!!}\frac{(n-5)!!}{(n-4)!!}\cdots\frac{(1)!!}{(2)!!}\frac{(0)!!}{(1)!!}\right] & \text{for odd n} \end{cases}$$

which is a telescoping product which simplifies nicely as,

$$\prod_{k=1}^{n-3}\int_0^{\pi}\sin^n(x)\ dx = \begin{cases} 2(2\pi)^{(n-4)/2}\frac{1}{(n-3)!!} & \text{for even n} \\ (2\pi)^{(n-3)/2}\frac{1}{(n-3)!!} & \text{for odd n} \end{cases} \qquad \textbf{Equation 22.}$$

Putting together **Equations 12, 16, and 22**, we get that for even n

$$C = \frac{2\pi R^{n+1}n(\frac{n}{2}-1)!}{(n+1)\pi^{n/2}R^n}\left(\frac{1}{n-1}\right)\left(2(2\pi)^{(n-4)/2}\frac{1}{(n-3)!!}\right) = \frac{2^{n/2}}{\pi}\frac{n(\frac{n}{2}-1)!}{(n+1)!!}\ R, \qquad \textbf{Equation 23.}$$

and if n is odd, we find that,

$$C = \frac{2\pi R^{n+1}(n)!!}{(n+1)(2\pi)^{[(n-1)/2]}R^n}\left(\frac{1}{n-1}\right)\left((2\pi)^{(n-3)/2}\frac{1}{(n-3)!!}\right) = \frac{n!!}{(n+1)!!}\ R \qquad \textbf{Equation 24.}$$

Thus, the centroid of an n-dimensional half ball of radius R can be described with the following (*piecewise*) sequence which depends only on the parity of n,

$$C_n = \begin{cases} \frac{2^{n/2}}{\pi}\frac{n(\frac{n}{2}-1)!}{(n+1)!!}\ R & \text{for even n} \\ \frac{n!!}{(n+1)!!}\ R & \text{for odd n} \end{cases} \qquad \textbf{Equation 25.}$$

Notice that since the centroid is a function of n we now add the subscript to C_n to specify the dimension. Overall this is a very interesting result but there are some questions which remain. For example notice that the value of C_n is much more concise for odd values of n than for even values of n. This suggests that there may be a more elegant way to express C_n for even n. If it exists, what is this expression? Additionally, one of the benefits of having a closed form expression for C_n as a function of n is that we may now test the global properties of the sequence. Specifically, does the sequence converge? If so, what does it converge to? The following sections work to find an alternate expression for the even subsequence and to test C_n in general for convergence.

Simplifying the Even Case

Let G_n be the even subsequence (C_n for even n). Then for all even integers $n>0$, G_n is defined by **Equation 26**,

$$G_n = \frac{2^{n/2}}{\pi}\frac{n(\frac{n}{2}-1)!}{(n+1)!!} \qquad \textbf{Equation 26.}$$

Consider the value of G_{n+2}. By definition we have,

$$G_{n+2} = \frac{2^{\frac{n}{2}+1}(n+2)(\frac{n}{2})!}{\pi(n+3)!!}$$
$$G_{n+2} = \frac{2 * 2^{n/2}(n+2)\frac{n}{2}(\frac{n}{2}-1)!}{\pi(n+3)(n+1)!!}$$
$$G_{n+2} = \frac{n+2}{n+3}\frac{2^{n/2}n(\frac{n}{2}-1)!}{\pi(n+1)!!}$$
$$G_{n+2} = G_n\frac{n+2}{n+3}$$

Thus, there is a recursive relation describing G_n. Just as we did before in determining the value of $\int_0^\pi \sin^n(x)\ dx$, we can repeatedly apply the recursive relation to relate G_n to the base case G_2 which has a known numerical value, meaning we once again have a closed form expression for G_n. In general, for any even positive integer n,

$$G_n = \frac{2}{\pi}\frac{n!!}{(n+1)!!} \quad \text{Equation 27.}$$

Therefore, we can express the even sequence in a much more elegant way, matching the form of the odd sequence. This gives us the revised definition for C_n,

$$C_n = \begin{cases} \frac{2}{\pi}\frac{n!!}{(n+1)!!}\ R & \text{for even n} \\ \\ \frac{n!!}{(n+1)!!}\ R & \text{for odd n} \end{cases} \quad \text{Equation 28.}$$

This alternative form of expressing the even sequence makes proving that C_n converges a much simpler task. Since the even and odd sequence only differ by a numerical factor, if we can show that the odd sequence $\frac{n!!}{(n+1)!!}$ is convergent then proving that the even sequence converges is trivial.

The Convergence of C_n

In this section we discuss the convergence of our sequence, C_n. Note that since C_n is a monotonic, decreasing sequence, bounded below by zero, then it must be that C_n converges to *something*. Then proving convergence alone is not too difficult of a problem. The true goal of this section is to show, specifically, that C_n converges to zero. Then let $U_n = \frac{n!!}{(n+1)!!}$ be the subsequence of C_n for all odd n.

Proposition 1. U_n converges to zero as n goes to infinity.

Proof. By definition, $\lim_{n\to\infty} U_n = 0$ if for any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left|\frac{n!!}{(n+1)!!}\right| < \varepsilon.$$

Since U_n is positive for all n then this requirement is the same as

$$\frac{n!!}{(n+1)!!} < \varepsilon,$$

or

$$\frac{(n+1)!!}{n!!} > \frac{1}{\varepsilon}.$$

In words, since U_n is always positive, to show that it converges to zero is equivalent to showing that the inverse is not bounded above.

Let $a_n = \frac{1}{U_n} = \frac{(n+1)!!}{n!!}$ be a sequence defined for positive odd integers, n. We can express this using product notation as follows,

$$a_n = \prod_{k=0}^{\frac{n-1}{2}} \frac{2k+2}{2k+1}.$$

With some algebra we find,

$$\begin{aligned} a_n &= \prod_{k=0}^{\frac{n-1}{2}} \frac{2k+2}{2k+1} \\ &= \prod_{k=0}^{\frac{n-1}{2}} \left(1 + \frac{2k+2}{2k+1} - 1\right) \\ &= \prod_{k=0}^{\frac{n-1}{2}} \left(1 + \frac{1}{2k+1}\right) \end{aligned}$$

By Theorem 3 of Section 28 in Knopp,[11] we know that in the limit as $n \to \infty$, a_n converges if and only if $\lim_{n\to\infty} \sum_{k=0}^{\frac{n-1}{2}} \frac{1}{2k+1}$ converges. By the integral test, the series $\sum_{k=0}^{\frac{n-1}{2}} \frac{1}{2k+1}$ is clearly divergent and so a_n must be divergent as well. Therefore, since a_n is not bounded above, then U_n converges to zero as $n \to \infty$. QED

The above proof only deals with the the odd sequence, U_n, however since G_n has the same form, the same process shows that G_n converges to zero as well. Therefore, we have shown that as $n \to \infty$, $C_n \to 0$.

DISCUSSION

This result leads to many interesting points. First, notice that due to the symmetry, the centroid of the n-ball is simply the origin regardless of the dimension. Then from the previous sections we see that as the dimension increases, the centroid of the half n-ball approaches the centroid of the n-ball. In other words, the higher the dimension of the n-ball, the more the mass becomes concentrated near the origin.

The case of the half n-ball is quite interesting, yet it leads to some natural following questions. In this project we focus solely on the case of uniform density, meaning the mass integral can be changed to a volume integral. It is interesting to consider what would happen to the centroid if the density depended on the radius rather than staying constant. Suppose the density took the form $\rho(r) = r^k$, where k is an integer. If k is negative then the mass concentration increases as you approach the origin and we expect that the COM would converge to zero at a faster rate. However if $k > 1$ it could lead to some interesting cases. For example, for $k \gg 1$, you can approximate the ball as if all the mass is distributed throughout the surface, and rather than dealing with the n ball we would be dealing with its outer shell. Additionally, is there some sort of critical mass distribution for which C_n does not converge as $n \to \infty$? Furthermore, all these questions deal with spherical geometries. What sort of trends are present when we consider the higher dimensional analogs of other shapes? Do we still expect their COMs to converge to the origin in higher dimensions. Consider the hypercube. Regardless of the dimension, the distance from the hyper cube's COM to the origin is the same. Could there be some shape for which the COM moves farther from the origin as we increase the dimension? Clearly, many related questions remain, and in the future it would be interesting to explore these alternative density distributions and geometries.

CONCLUSIONS

This project worked to generalize the problem of the COM of the half disk (2D) and half ball (3D) to the arbitrary n-dimensional case with the goal of both generalizing our 2D and 3D solutions as well as gaining more understanding as to why we see the trend we do at lower ($n \leq 3$) dimensions. In order to solve the problem in n dimensions we leveraged the spherical symmetry of the system just as in the lower dimensional cases. First, we chose the basis so that the axis of symmetry lies along only one of the Cartesian axes. This allowed us to describe the COM vector using only one component which simplifies calculations substantially. After choosing the correct coordinate system, solving the integral was possible with a few intermediate steps. As a result we derived a function for the COM which depends purely on the number of dimensions n. By finding a closed form function describing the centroid of the half n-ball, not only do we construct a more general solution but we also are now able to test the global properties of the function as we vary n. The ability to explore trends in an *analytic* way gives us a much deeper insight to the problem. In particular, we show that $\lim_{n\to\infty} C_n = 0$.

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PRESS SUMMARY

This project explores the center of mass (COM) of a certain class of objects in different dimensions. The COM is an important property in physics used to determine the motion of objects as it drastically simplifies calculations. In particular, we study the half n-dimensional ball, for example the half disk ($n = 2$), half ball ($n = 3$), and their higher ($n > 3$) dimensional analogues. In our study, we let the density of the object be constant throughout so that the COM is the same as the physical center of the object, known as the centroid. Rather than performing manual computations for each individual case we find a formula describing the centroid in any dimension. Furthermore, we analyze the trend as the number of dimensions increases. We show that as the number of dimensions increases infinitely, the centroid of the *half* n-ball gets arbitrarily close to centroid of the *full* n-ball.