# Finding the Fixing Number of Johnson Graphs $J(n, k)$ for $k \in \{2, 3\}$

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**ABSTRACT**
The graph invariant, aptly named the fixing number, is the smallest number of vertices that, when fixed, eliminate all non-trivial automorphisms (or symmetries) of a graph. Although many graphs have established fixing numbers, Johnson graphs, a family of graphs related to the graph isomorphism problem, have only partially classified fixing numbers. By examining specific orbit sizes of the automorphism group of Johnson graphs and classifying the subsequent remaining subgroups of the automorphism group after iteratively fixing vertices, we provide exact minimal sequences of fixed vertices, in turn establishing the fixing number of infinitely many Johnson graphs.

**KEYWORDS**
Graph Automorphism Groups; Symmetry Breaking; Fixing Number; Determining Number; Johnson Graphs; Kneser Graphs; Graph Invariants; Permutation Groups; Minimal Sized Bases

**INTRODUCTION**
Since Euler first posed the Seven Bridges of Königsberg problem in 1736, graph theory has rapidly evolved into a prominent field of research in mathematics. Deceivingly simple, graphs provide a powerful mathematical description of the relationships between elements of a system. Graphs may be used to represent physical structures such as organic molecules, electrical circuits, and road systems, in addition to non-physical networks like social groups, data structures, and linguistics.[1–5]

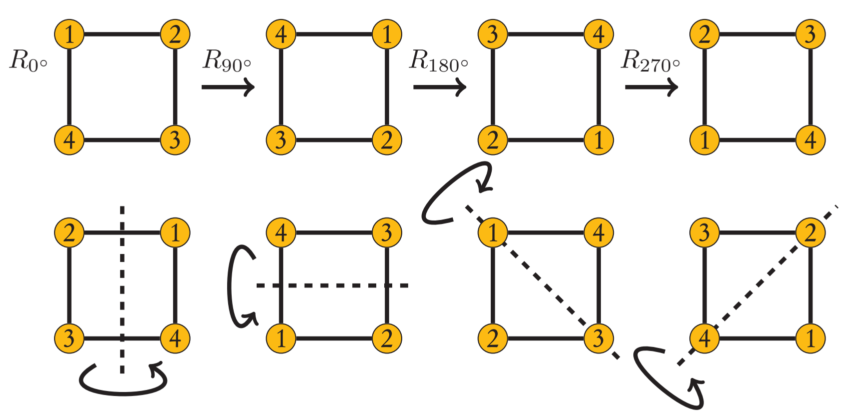



**Figure 1.** A 2-regular graph on four vertices and four edges has 8 automorphisms, with an automorphism group isomorphic to the dihedral group $D_4$.

A graph $\Gamma = (V, E)$ is an unordered double composed of a non-empty vertex set $V$ and an edge set $E$, consisting of unordered pairs of vertices from $V$. A vertex $v_i$ is adjacent to the vertex $v_j$ if and only if there exists an edge $v_i \sim v_j \in E$. The degree of a vertex $v$ is the number of edges containing $v$, and a $k$-regular graph is defined by all vertices having equal degree $k$. Only simple graphs with one connected component are considered in this paper (except for the null graphs in **Proposition 2**), meaning undirected edges with two distinct vertices and at most one edge between vertices.

An automorphism, or symmetry, of a graph $\Gamma$ is a bijection $\phi : V \to V$ that preserves the adjacency relations of $E$ such that $\phi(v_i) \sim \phi(v_j) \in E$ if and only if $v_i \sim v_j \in E$. The set of all automorphisms of $\Gamma$ forms a permutation group on the vertex set $V$, called the automorphism group of $\Gamma$, and is denoted by $\text{Aut}(\Gamma)$.[6] **Figure 1** presents all eight automorphisms of a 2-regular graph on four vertices and four edges. It should be noted that not all graphs have automorphism groups consisting of different movements. In fact, Erdős showed that the majority of graphs possess only the identity automorphism; such graphs are fittingly named rigid graphs.[7]

We now present a graph invariant that quantifies the complexity of a graph. The fixing number of a graph $\Gamma$, denoted $\text{fix}(\Gamma)$, is the minimum number of vertices that, when fixed, eliminate all nontrivial automorphisms of the graph $\Gamma$. This minimal sized vertex subset produces a compact representation of the graph's structure, since any of its automorphisms can be characterized by its action on at least one of these fixed vertices. Also known as symmetry breaking, eliminating graph automorphisms has numerous applications such as characterizing the synthesis of amino acids or assisting robotic systems in determining orientation.[8, 9] This topic has also been investigated under different guises; the determining number or metric dimension of a graph, or, in a purely algebraic view, a minimal sized base of a permutation group acting faithfully on a set.[10]

We now state a few results on established fixing numbers for certain graph families. For ease of notation, let $[m, n]$ represent the set $\{m, m+1, m+2, \ldots, n-1, n\}$ for $m, n \in \mathbb{Z}$ and $m < n$; the set $\{1, \ldots, n\}$ is denoted as $[n]$.

**Proposition 1.** *For a cycle graph $C_n$ with $n \geq 3$, fix $(C_n) = 2$.*

*Proof.* A cycle graph on $n$ vertices is defined as $C_n = ([n], \{i \sim j$ if and only if $j - i \equiv 1 \; mod \; n$ for all $i, j \in [n],\ i \neq j\})$, with the automorphism group $\text{Aut}(C_n)$ isomorphic to the dihedral group $D_n$. The 2-regular graph in **Figure 1** is an example of the cycle graph on 4 vertices $C_4$. Fixing any single vertex results in an automorphism group isomorphic to $\mathbb{Z}_2$, since a reflection across the longest line of symmetry through the fixed vertex and the face of the graph remains. Therefore, fixing any adjacent vertex to the previously fixed vertex will produce a trivial automorphism group. See **Figure 2** for an example. □

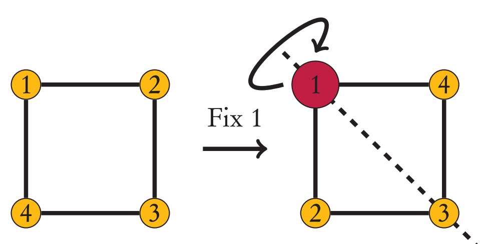



**Figure 2.** Fixing the vertex labeled 1 leaves a reflection such that the vertices 2 and 4 may be interchanged. Subsequently, fixing the vertex labeled 2 eliminates all nontrivial automorphisms of the graph.

**Proposition 2** illustrates that a graph's connectedness might not impact the determination of the fixing number.

**Proposition 2.** *The complete graph $K_n$ and the null graph $N_n$ on $n$ vertices have fixing number equal to $n - 1$.*

*Proof.* The complete and null graphs are defined as $K_n = ([n], \{i \sim j$ for all $i, j \in [n],\ i \neq j\})$ and $N_n = ([n], \emptyset)$ respectively. In either graph, any vertex can be mapped to any other vertex, and therefore the total number of automorphisms is $n!$ with both automorphism groups isomorphic to $S_n$. Fixing one vertex eliminates exactly $n$ automorphisms and leaves an automorphism group isomorphic to $S_{n-1}$. Therefore, fixing $n-1$ vertices results in a trivial automorphism group since the 'free' unfixed vertex cannot be interchanged with any other vertex, and therefore is also fixed. □

*Johnson Graphs*

The focus now shifts to determining the fixing numbers of an infinite sized subset of Johnson graphs, a two parameter graph family denoted as $J(n, k)$.

**Definition 1** ([6, 11]). For positive integers $n$ and $k$, the Johnson graph $J(n, k)$ for $2 \leq k < \lfloor \frac{n}{2} \rfloor$ is defined as

1. $V(J(n, k))$ consists of unordered $k$-tuples of $[n]$.
2. The edge $v_i \sim v_j \in E(J(n, k))$ if and only if $|v_i \cap v_j| = k - 1$, *i.e.,* the number of integers in the intersection of the vertex labels equals $k - 1$.

The following properties can be derived from the definition.

- $|V(J(n, k))| = \binom{n}{k}$.
- $\text{Aut}(J(n, k)) \cong S_n$.
- Johnson graphs are *vertex-transitive*, meaning that $\forall v_i, v_j \in V(J(n, k)),\ \exists \sigma \in \text{Aut}(J(n, k))$ such that $\sigma(v_i) = v_j$.
- The distance between any two vertices $d(v_i, v_j)$ remains constant under any $\sigma \in \text{Aut}(J(n, k))$, implying Johnson graphs are *distance-transitive*.

The combinatorial identity $\binom{n}{k} = \binom{n}{n-k}$ explains the restriction $k \in \left[2, \frac{n}{2}\right)$ since $J(n, k) \cong J(n, n-k)$ for $k > \frac{n}{2}$. For $k = 1$, the resulting graph is the complete graph $K_n$ defined in **Proposition 2**, of which the fixing number was shown. The second condition of **Definition 1** describes the edge set of $J(n, k)$. However, *generalized* Johnson graphs $J(n, k, i)$ specify that vertices are adjacent if the intersection of integers in vertex labels is equal to $i \in [0, k-1]$, of which $i = 0$ defines what are known as Kneser graphs.[6] While the fixing numbers of Kneser graphs have attracted much attention (under the name determining number), the flexibility in constructing edge sets provided by generalized Johnson graphs allows us to classify the fixing number for a significantly larger number of graphs. We provide a few established results on fixing numbers of Kneser graphs; notation has been adjusted for consistency.

Caceres *et al.* found the fixing number for a large subset of Kneser graphs through the following two theorems.

**Theorem 1** (Caceres et al.[12]). *Let $k, d \in \mathbb{Z}^+$ such that $k \leq d$ and $d > 2$. If $n = \left\lfloor \frac{d(k+1)}{2} \right\rfloor + 1$, then*

$$\textit{fix}(J(n, k, 0)) = d.$$

**Theorem 2** (Caceres et al.[12]). *Let $k, d \in \mathbb{Z}^+$ such that $3 \leq k + 1 \leq d$. For all $n \in \mathbb{N}$ such that $\left\lfloor \frac{(d-1)(k+1)}{2} \right\rfloor < n < \left\lfloor \frac{d(k+1)}{2} \right\rfloor$, then*

$$\textit{fix}(J(n + 1, k, 0)) = d.$$

It should again be emphasized that establishing the fixing number of Johnson, Kneser, or generalized Johnson graphs are equivalent provided that the two parameters $n$ and $k$ agree.

Though beyond the scope of this paper, Babai's recent work on reducing the graph isomorphism problem to quasi-polynomial time illustrated the importance of Johnson graphs, which he described as "a source of just unspeakable misery"[13, 14]. It is not difficult to see why these graphs pose such a problem; they are $k(n-k)$-regular with distance preserved under any of the $n!$ automorphisms. The performance improvement of his algorithm stems from the recognition of whether a graph contains a Johnson graph as a subgraph and employing the appropriate methods upon that decision. Babai's own words reinforce the motivation for our work: "In fact, breaking regularity is one of the key tools in the design of algorithms for graph isomorphism; the graph isomorphism problem has therefore been one of the strongest motivators of the study of all sorts of 'resolving/discriminating sets', and perhaps the only deep motivator of the study of those in contexts where no group is present."[10]

*Fixing Johnson Graphs*

In this section, we establish the fixing number of the Johnson graphs $J(n,k)$ for $n > 2k$ and $k \in \{2,3\}$. Throughout, we assume for a finite graph $\Gamma = J(n,k)$ the following action of $\text{Aut}(\Gamma)$ on $V(\Gamma)$, which is the standard action of $S_n$ acting on $k$-subsets.

**Definition 2.** Let $\Gamma$ be a graph such that $\text{Aut}(\Gamma) \cong S_n$ acts transitively on the unordered $k$-subsets of $[n]$ such that for all $\sigma \in S_n$ and $\forall\{\ell_1, \ell_2, \ldots, \ell_k\} \in V(\Gamma)$,

$$\sigma \cdot \{\ell_1, \ell_2, \ldots, \ell_k\} = \{\sigma(\ell_1), \sigma(\ell_2), \ldots, \sigma(\ell_k)\}.$$

Since Johnson graphs are vertex-transitive, there exists only one orbit in $V(\Gamma)$ of size $\binom{n}{k}$.

**Theorem 3.** *If* $\Gamma = J(n,2)$ *where* $n \geq 5$, *then fix*$(\Gamma) = \left\lfloor \frac{2n}{3} \right\rfloor$.

*Proof.* Let $\text{Aut}(\Gamma) \cong S_n$ act transitively on the 2-subsets of $V(\Gamma)$ by the action specified in **Definition 2**. Consider fixing the vertex with integer label $\{1,2\}$ so that it may not be permuted among any other vertices in $V(\Gamma)$. The remaining non-trivial elements of the automorphism group have exactly two forms

$$(1,2)\tau \quad \text{or} \quad (1)(2)\tau \tag{1}$$

where $\tau$ is any permutation of $[3,n]$. The transposition $(1,2)$ remains since $\{1,2\} \equiv \{2,1\}$, *i.e.,* the vertex is fixed, but the integers 1 and 2 may still permute in the fixed vertex label when the automorphism group acts on $V(\Gamma)$. Therefore, fixing the vertex with integer label $\{1,2\}$ results in a subgroup of automorphisms isomorphic to $S_{n-2} \times \mathbb{Z}_2$.

Now, fix another element of $V(\Gamma)$, the vertex with integer label $\{2,3\}$. Every element that had the form $(1,2)\tau$ is now eliminated, since the integers 1 and 2 are no longer able to permute without also moving the now fixed vertex $\{2,3\}$. In this sense, the approach can be viewed as finding the optimal method of fixing integers $[n-1]$ in the vertex labels of elements in $V(\Gamma)$ by correctly choosing the appropriately labeled vertices. A crucial observation of this work is the attempt to only fix the integers $[n]$ if necessary, otherwise fixing $[n-1]$ achieves the desired result and may result in a smaller fixing number. The remaining non-trivial elements of the automorphism group are represented as

$$(1)(2)(3)\rho$$

where $\rho$ is any permutation of the integers $[4,n]$. An algorithmic approach is to continue fixing vertices with integer labels in a sequence of at most

$$\{1,2\}, \{2,3\}, \{4,5\}, \{5,6\}, \{7,8\}, \{8,9\}, \ldots, \{n-2, n-1\}, \{n-1, n\}. \tag{2}$$

Begin by examining the smallest unfixed consecutive integers $x, y$, and $z$ in the vertex labels of $V(\Gamma)$. At each step in the algorithm, if the vertex label $\{x,y\}$ consists of integers of which neither has been fixed in a previous step, then the automorphism group that remains when fixing $\{x,y\}$ is isomorphic to $S_{n-y} \times \mathbb{Z}_2$. However, if $y$ has already appeared in a previously fixed vertex label, then by fixing the next vertex label $\{y, z\}$, the resulting automorphism group will be isomorphic to $S_{n-z}$. We deviate from this algorithm only if $n \equiv 2 \ mod \ 3$, at which the last 3 fixed vertices have the form

$$\{x, y\}, \{y, z\}, \{z, z+1\}. \tag{3}$$

By fixing pairs of vertices with consecutive integer labels $\{x, y\}$ and $\{y, z\}$, the resulting automorphism group is isomorphic to $S_{n-3}$ since the integers $x, y$, and $z$ in the vertex labels of $V(\Gamma)$ are fixed.

To show the upper bound of fix$(\Gamma) \leq \lfloor \frac{2n}{3} \rfloor$, write $n = 3q + r$ for $q = \lfloor \frac{n}{3} \rfloor$ and $r \equiv n \bmod 3$. By noting that each pair of fixed vertices of the form $\{x, y\}$ and $\{y, z\}$ fixes the three integers $x, y$, and $z$ in the vertex labels of $V(\Gamma)$, we consider two cases.

- If $r = 0$, a fixing set $F$ can be given as

$$F = \bigcup_{i=1}^{q} \{\{3i-2, 3i-1\}, \{3i-1, 3i\}\} \tag{4}$$

  resulting in a fixing set of size $|F| = 2q$. If $r = 1$, then $F$ fixes the integers $[n-1]$ in the vertex labels of $V(\Gamma)$, stabilizing the integer $n$.
- If $r = 2$, then the last three fixed vertex labels are defined in **(3)**. Therefore, fixing $2q$ vertices given by **(4)** and one additional vertex yields

$$F' = F \cup \{\{3q, 3q+1\}\}.$$

In either case, we have that $|F| = |F'| = \lfloor \frac{2n}{3} \rfloor$ which shows fix$(\Gamma) \leq \lfloor \frac{2n}{3} \rfloor$. To show fix$(\Gamma) \geq \lfloor \frac{2n}{3} \rfloor$, we build upon a combinatorial argument that appeared in Tracey Maund's unpublished dissertation on base sizes of permutation groups.[15] For ease of notation, define $f = $ fix$(\Gamma)$ for a fixing set $F^*$. Clearly, $n = \sum_{i=0}^{f} n_i$ where $n_i$ is the number of occurrences that an integer is present in $i$ vertices of the fixing set. Similarly, $f \cdot k$ is the total number of places that an integer can occupy in the vertices of a fixing set, since there are $f$ vertices each of size $k$. So,

$$\begin{aligned}
f \cdot k &= \sum_{i=0}^{f} i \cdot n_i \\
&= 0 \cdot n_0 + 1 \cdot n_1 + 2 \cdot n_2 + \cdots + f \cdot n_f \\
&= 2 \cdot (n_1 + n_2 + \cdots + n_f) - n_1 + [n_3 + 2 \cdot n_4 + \cdots + (f-2) \cdot n_f] \\
&= 2 \cdot (n - n_0) - n_1 + [n_3 + 2 \cdot n_4 + \cdots + (f-2) \cdot n_f] \\
&\geq 2\,(n - n_0) - n_1.
\end{aligned}$$

Let $n_0 \leq 1$ since $n_0 \geq 2$ implies the transposition $(1, 2) \in$ Aut$(\Gamma)$, contradicting that $F^*$ is a fixing set. Next, note that $n_i \leq f$, since no integer can appear in more than $f$ vertices. Then

$$\begin{aligned}
f \cdot k \geq 2(n - n_0) - n_1 &\geq 2(n-1) - f \\
f &\geq \frac{2(n-1)}{k+1}.
\end{aligned}$$

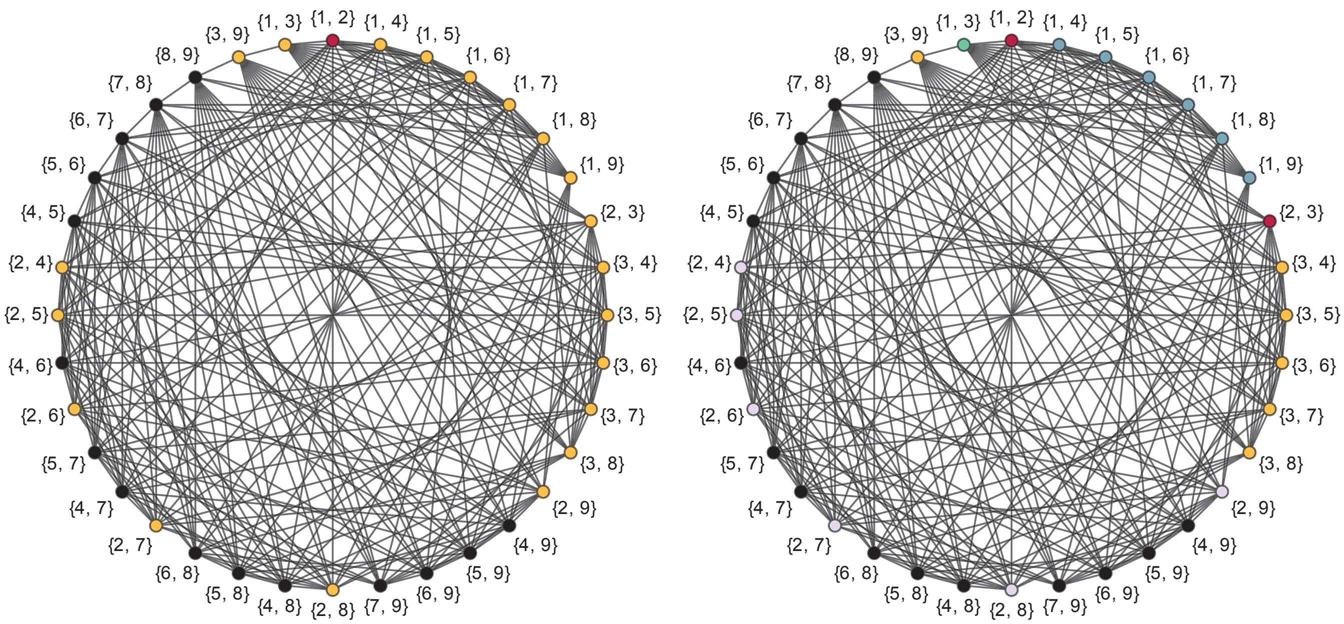
Since $f \in \mathbb{N}$, then $f \geq \left\lceil \frac{2(n-1)}{k+1} \right\rceil$. In general, for $s, t \in \mathbb{Z}$, $\left\lceil \frac{s}{t+1} \right\rceil = \left\lfloor \frac{s+t}{t+1} \right\rfloor$, and so $\left\lceil \frac{2(n-1)}{k+1} \right\rceil = \left\lfloor \frac{2(n-1)+k}{k+1} \right\rfloor$. For $k = 2$, this reduces down to

$$\begin{aligned}
f &\geq \left\lfloor \frac{2(n-1)+2}{2+1} \right\rfloor \\
\text{fix}(\Gamma) &\geq \left\lfloor \frac{2n}{3} \right\rfloor.
\end{aligned}$$

Therefore, fix$(\Gamma) = \lfloor \frac{2n}{3} \rfloor$. □

We now provide an example in order to classify the subgroups of the automorphism group that remain when iteratively fixing vertices and the algorithmic approach for choosing specific vertex labels.

**Example 1.** Consider the Johnson graph $J(9,2)$ which has unordered doubles of $[1,9]$ as vertices and edges $v_i \sim v_j$ if the respective vertex labels share exactly one integer entry, seen in **Figure 3**. First, fix the vertex with integer label $\{1,2\}$. All remaining non-trivial elements in the automorphism group have the form $\psi$ or $(1,2)\psi$ where $\psi$ is any permutation of the integers $[3,9]$, indicating a subgroup of automorphisms isomorphic to $\mathbb{Z}_2 \times S_7$. Vertices with labels containing only the integers $[3,9]$ can freely permute, but vertices with integer labels $\{i,j\}$ for $i \in [1,2]$ and $j \in [3,9]$ now form a separate orbit of size 14, illustrated by the gold vertices seen in **Figure 3a**. Proceed by fixing the vertex with the integer label $\{2,3\}$. Every remaining non-trivial element $\nu$ of the automorphism group consists of permutations of $[4,9]$, since the transposition $(1,2)$ can no longer act on $V(\Gamma)$ without permuting the vertex labeled $\{2,3\}$. Although only two fixed vertices have been chosen, now the vertex with label $\{1,3\}$ is also fixed as a result. Any of the gold, blue, or pink vertices seen in **Figure 3b** with integer labels $\{i,j\}$ for $i \in [1,3]$ and $j \in [4,9]$ may only permute among vertices of the same color (*i.e.*, sharing the same $i$ value; black vertices may still be mapped freely), indicating the remaining automorphism group is isomorphic to $S_6$.




**(a)** Fixing the vertex $\{1,2\}$ reduces the total number of automorphisms from 362,880 to 10,080.

**(b)** Fixing the red vertices with integer labels $\{1,2\}$ & $\{2,3\}$ consequentially fixes the green vertex with label $\{1,3\}$.

**Figure 3.** Fixing two vertices of $J(9,2)$ with integer labels $\{1,2\}$ & $\{2,3\}$ reduces the size of the automorphism group from $362,880$ to $720$, a $99.8\%$ decrease.

Following the sequence of fixed vertices given by **(2)**, fixing vertices with integer labels

$$\{1,2\},\{2,3\},\{4,5\},\{5,6\},\{7,8\},\{8,9\},$$

yields the associated chain of subgroups of $S_9$

$$S_7 \times \mathbb{Z}_2 \geq S_6 \geq S_5 \times \mathbb{Z}_2 \geq S_4 \geq S_3 \times \mathbb{Z}_2 \geq \mathbb{Z}_2 \geq \{e\}.$$

Therefore, the fixing number of $J(9,2)$ is equal to $\left\lfloor \frac{2(9)}{3} \right\rfloor = 6$, since any subset of 5 fixed vertices will leave at least one remaining transposition in the automorphism group.

We now utilize much of the same machinery from the proof of **Theorem 3** to establish the fixing numbers of Johnson graphs $J(n,3)$ for $n \geq 7$.

**Theorem 4.** *If* $\Gamma = J(n,3)$ *where* $n \geq 7$*, then fix*$(\Gamma) = \lfloor \frac{n}{2} \rfloor$.

*Proof.* Let $\text{Aut}(\Gamma) \cong S_n$ act transitively on the 3-subsets of $V(\Gamma)$ by the action stated in **Definition 2** and consider fixing the vertex with integer label $\{1,2,3\}$. The remaining non-trivial elements of the automorphism group have the following 6 forms,

$$(1,2)\tau \quad (1,3)\tau \quad (2,3)\tau \quad (1,2,3)\tau \quad (1,3,2)\tau \quad \tau \tag{5}$$

where $\tau$ is any permutation of $[4,n]$. Therefore, the subgroup of automorphisms consisting of all forms in **(5)** and the identity automorphism is isomorphic to $S_3 \times S_{n-3}$.

Fix another vertex with the integer label $\{3,4,5\}$. Every element of $S_n$ that had a cycle containing the integer 3 is now eliminated, since each of the previous two fixed vertex labels contained 3. The remaining non-trivial elements in the automorphism group each have one of the following 7 forms,

$$(1,2) \quad (4,5) \quad (1,2)(4,5) \quad \rho \quad (1,2)\rho \quad (4,5)\rho \quad (1,2)(4,5)\rho$$

for any permutation $\rho$ of the integers $[6,n]$. Subsequently, the remaining subgroup of the automorphism group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_{n-5}$. Next, fix the vertex with integer label $\{5,6,7\}$, in turn fixing the integers $[3,5]$ and resulting in an automorphism group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_{n-7}$ (the transpositions $(1,2)$ and $(6,7)$ remain). Continue fixing vertices of consecutive integers that have the first and last integers appearing in some other fixed vertex label. However, note that the integer 1 needs to appear in the last fixed vertex label chosen to eliminate the remaining transposition $(1,2)$.

The general approach is to proceed by fixing vertices with integer labels in a sequence of at most,

$$\{1,2,3\},\{3,4,5\},\{5,6,7\},\{7,8,9\},\ldots,\{n-5,n-4,n-3\},\{n-3,n-2,n-1\},\{n-1,n,1\}.$$

Start by examining the smallest unfixed consecutive integers $s, t$, and $u$ in the vertex labels of $V(\Gamma)$ (note that $s$ has appeared in a previously fixed vertex label). Fixing the vertex with label $\{s,t,u\}$ results in an automorphism group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_{n-u}$. The next fixed vertex will have an integer label of the form $\{u,v,x\}$ for consecutive integers $u, v$, and $x$ and produces an automorphism group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_{n-x}$, since the transpositions $(1,2)$ and $(v,x)$ remain but the integer $u$ is fixed. Fixing another vertex with integer label $\{x,y,z\}$ eliminates one transposition $(v,x)$ but introduces the transposition $(y,z)$, and so the automorphism group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_{n-z}$. Similar to **Theorem 3**, observe that fixing $3r$ vertices fixes $6r-3$ integers in the vertex labels of $V(\Gamma)$.

Now, write $n = 2q + s$ for $q = \lfloor \frac{n}{2} \rfloor$ and $s \equiv n \bmod 2$. If the parity of $n$ is even, then the last vertex to fix will have the form $\{n-1,n,1\}$ and a fixing set of size equal to $\frac{n}{2}$. For $n$ odd, the last vertex will have the integer label $\{n-2,n-1,1\}$, and consequently the fixing set will be of size $\frac{n-1}{2}$. In either case, the fixing set $F$ can be given by

$$F = \{\{2q-1,2q,1\}\} \cup \bigcup_{i=1}^{q-1} \{2i-1,2i,2i+1\},$$

and so $\text{fix}(\Gamma) \leq \lfloor \frac{n}{2} \rfloor$.

To show the opposite inequality, recall $n = \sum_{i=0}^{f} n_i$ where $n_i$ is the number of integers that appear $i$ times in the vertices of a fixing set. Since $f \cdot k$ is the number of entries integers can occupy in the vertices of the fixing set then, similar to **Theorem 3**, we have $f \cdot k \geq 2(n - n_0) - n_1$. Since $n_i \leq f$ we choose $n_1 = f$ to be of maximal value, but now consider two cases based upon $n_0 \equiv n \bmod 2$.

- For $n_0 = 0$,

$$\begin{aligned} f \cdot k &\geq 2n - f \\ &= \left\lceil \frac{2n}{k+1} \right\rceil \\ &= \left\lceil \frac{2n}{4} \right\rceil \\ &= \left\lceil \frac{n}{2} \right\rceil. \end{aligned}$$

  Since $n$ is even, then $\left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$.

- For $n_0 = 1$,

$$\begin{aligned} f \cdot k &\geq 2(n-1) - f \\ &= \left\lceil \frac{2(n-1)}{k+1} \right\rceil \\ &= \left\lceil \frac{2(n-1)}{4} \right\rceil \\ &= \left\lceil \frac{n-1}{2} \right\rceil. \end{aligned}$$

  Given that $n$ is odd, then $\left\lceil \frac{n-1}{2} \right\rceil = \left\lfloor \frac{n-1}{2} \right\rfloor$ which is equivalent to $\left\lfloor \frac{n}{2} \right\rfloor$.

In either case, we have that $\text{fix}(\Gamma) \geq \left\lfloor \frac{n}{2} \right\rfloor$ and therefore $\text{fix}(\Gamma) = \left\lfloor \frac{n}{2} \right\rfloor$. □

## CONCLUSIONS

In this work, the fixing number for an infinite sized subset of Johnson graphs was established by studying the action of the automorphism group on the vertex set, methodically choosing which vertices to fix by the associated integer labels, and observing the resulting automorphism groups. The fixing numbers calculated in **Theorems 3 & 4** unify the two separate results established by Caceres *et al.* in **Theorems 1 & 2** for specific values of $k$. Future work includes publishing fixing number results for an even greater subset of Johnson graphs for $k \geq 4$ and exploring the viability of applying similar number schemes in a cryptological system.

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James Della-Giustina graduated from Towson University with a B.S. in Spring ’22 and is slated to earn an M.S in Fall ’23; both in applied mathematics. After graduation, he hopes to gain a few years of industry experience before pursuing a Ph.D. focused in algebra and discrete structures.

## PRESS SUMMARY

We study a concept called the fixing number, which is the smallest number of vertices needed to eliminate all symmetries (mappings consisting of rotations, reflections, *etc.*) of a mathematical object called a graph. While many graphs have known fixing numbers, the fixing numbers of Johnson graphs - a family of graphs relevant to the study of graph isomorphism - are only partially understood. By iteratively fixing vertices and analyzing the sizes of resulting subgroups within the automorphism group, we are able to determine the fixing number for infinitely many Johnson graphs. Our findings have important implications for understanding the structure of these graphs and could have applications in fields such as computer science and cryptography.