On Packing Thirteen Points in an Equilateral Triangle

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ABSTRACT
The conversation of how to maximize the minimum distance between points - or, equivalently, pack congruent circles - in an equilateral triangle began by Oler in the 1960s. In a 1993 paper, Melissen proved the optimal placements of 4 through 12 points in an equilateral triangle using only partitions and direct applications of Dirichlet's pigeon-hole principle. In the same paper, he proposed his conjectured optimal arrangements for 13, 14, 17, and 19 points in an equilateral triangle. In 1997, Payan proved Melissen's conjecture for the arrangement of fourteen points; and, in September 2020, Joos proved Melissen's conjecture for the optimal arrangement of thirteen points. These proofs completed the optimal arrangements of up to and including fifteen points in an equilateral triangle. Unlike Melissen's proofs, however, Joos's proof for the optimal arrangement of thirteen points in an equilateral triangle requires continuous functions and calculus. I propose that it is possible to continue Melissen's line of reasoning, and complete an entirely discrete proof of Joos's Theorem for the optimal arrangement of thirteen points in an equilateral triangle. In this paper, we make progress towards such a proof. We prove discretely that if either of two points is fixed, Joos's Theorem optimally places the remaining twelve.

KEYWORDS
optimization; packing; equilateral triangle; distance; circles; points; thirteen; maximize

INTRODUCTION

Packing is a class of optimization problems. The objective is to place some number of non-overlapping geometric objects such that they are entirely contained in a larger object leaving as little space remaining in the larger object as possible. Packing problems have an expansive history beginning from when Kepler conjectured the density of the densest ball packing in 1611.[1] In this paper, we consider the problem of packing thirteen congruent circles in an equilateral triangle. It is important to note here the relationship between the problem of packing thirteen congruent circles into an equilateral triangle and maximizing the minimum distance between thirteen points in an equilateral triangle. As **Figure 1** below shows, the centers of the circles in the optimal packing of thirteen congruent circles in the larger triangle are the points that maximize the minimum distance between thirteen points in the smaller triangle. Thus, these are the same problem.

The two interpretations of the problem lead to different applications. By looking at the problem as a packing of circles, we answer questions along the lines of how large a box needs to be in order to hold n bottles of water. By considering the problem in terms of maximizing the distance between points on a plane, we solve problems like how to optimally place transistors onto a microchip. Throughout this paper, we will consider the problem in terms of maximizing the minimum distance between thirteen points in an equilateral triangle.

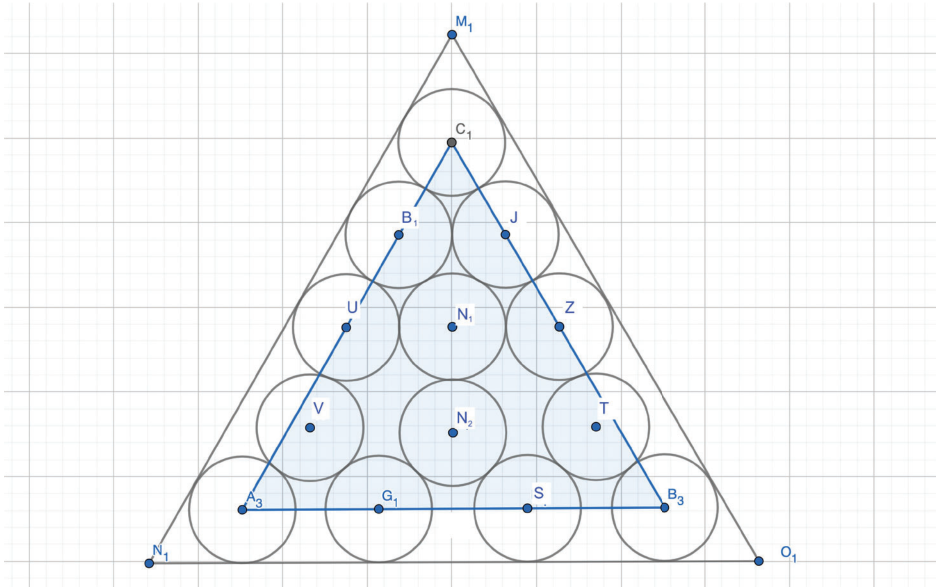

Figure 1. An equilateral triangle with optimal point packing within an equilateral triangle with optimal circle packing

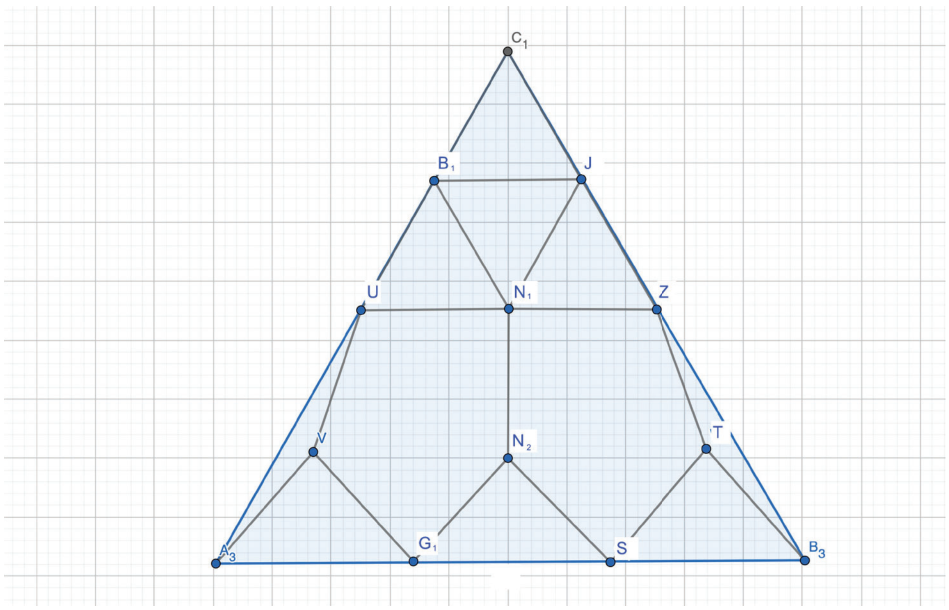

Figure 2. Optimal configuration of thirteen points in an equilateral triangle

Configurations that maximize the minimum distance between n points in an equilateral triangle have been proven for $n \leq 12$,[2, 3] $n = 14, 20$,[4] $n = k(k+1)/2$ for any $k \in N$, [5] and now $n = 13$.[6] $k(k+1)/2$ is the k^{th} triangular number, and the optimal packing is the obvious one: in rows of $1, 2, 3, ..., k$ points. The proof for $n = 13$ was completed by Joos in 2020, confirming Melissen's 1993 conjecture that the orientation of thirteen points in an equilateral triangle shown in **Figure 2** uniquely maximizes the minimum of the distance between two points.[2] In other words, no other configuration of thirteen points in an equilateral triangle exists such that the distance between every pair of points is more than or equal to the minimum distance between the points in **Figure 2**. We will denote this minimum distance as d_{13}.

Joos's proof for the optimal configuration of thirteen points in an equilateral triangle diverged from the strategies that had been used previously to find optimal configurations of points in equilateral triangles. Rather than using only discrete mathematics, his proof requires continuous functions and calculus, relies heavily on inequalities, and considers several cases. This paper explores the possibility of an entirely discrete proof of the optimal arrangement of thirteen points in an equilateral triangle. We demonstrate that if one can prove that an optimal placement of thirteen points must include N_1 or N_2 from **Figure 2** then we would have an alternate and discrete proof that the Melissen placement of points its optimal. Determining whether the methods that had worked for proving the optimal arrangements of fewer points are still valid for thirteen points could influence whether people must search for new approaches as the number of points continues to increase.

Joos computes $d_{13} = 9 - 5\sqrt{3} - \frac{7\sqrt{6}}{2} + 6\sqrt{2} \approx 0.251813$ for a triangle with side length 1.[6] Pairs of points in **Figure 2** above whose distance is d_{13} are connected by a grey line.

RESULTS

We know from Melissen's 1993 paper that the configuration of points shown in **Figure 2** exists such that pairs of points connected by a grey line in the figure are d_{13} apart.[2] From there, his conjecture states that this configuration is both optimal and unique, meaning that there is no other configuration of thirteen points in an equilateral triangle such that every pair of points is more than or equal to d_{13} apart. We will prove two theorems: one showing that if we assume the position of N_1 from **Figure 2**, then Joos's Theorem for thirteen points holds; and the other showing that if we assume the position of N_2, then Joos's Theorem holds as well.

We first note that Melissen proved the following lemma:

Lemma. In an optimal configuration of $n \geq 3$ points in an equilateral triangle, the three vertices of the triangle must be among the selected points.[3]

Theorem 1. *Let N_1 from **Figure 2** be the point that is d_{13} from each of the points on adjacent sides of the triangle that are d_{13} from the top corner. If N_1 is fixed, then Joos's Theorem for the optimal arrangement of 13 points is correct.*

In other words, given the position of N_1, there is no arrangement of the remaining twelve points in the triangle other than the arrangement shown in **Figure 2** such that every pair of two of the thirteen points is at least d_{13} apart. We first realize that for such a configuration to exist, it cannot contain any other points in the interior of a circle with radius d_{13} centered at N_1. As **Figure 3** shows, this creates two distinct regions: the region above the circle about N_1 and the region below. We will begin by examining the upper region.

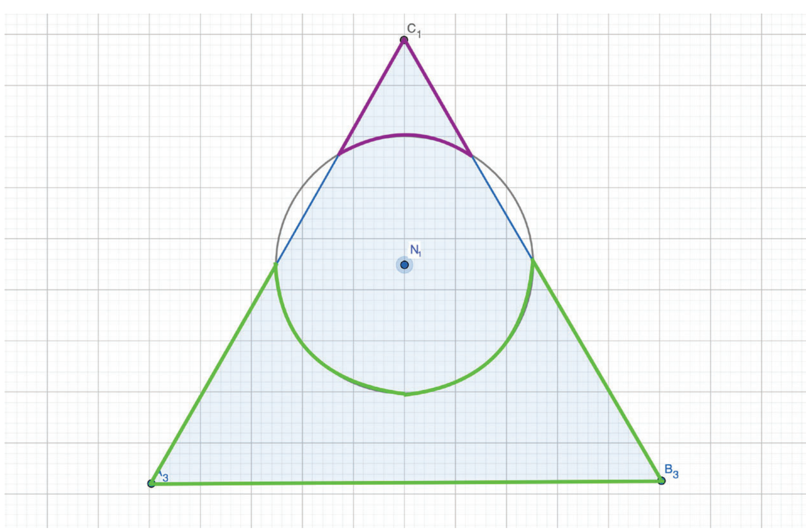

Figure 3. Equilateral triangle with a circle of radius d_{13} about N_1

Given our definition of N_1 and as **Figure 4** illustrates, the intersection of the interiors of circles of radius d_{13} about each of the three corners of the upper region contain every point except the three corners. Thus we know that every point other than the corners is less than d_{13} every other point in the region. So in order for there to be at least two points in this region whose distance apart is at least d_{13}, those points must be the corners. This also shows that we can include at most three points in this region. We can achieve this only by placing them in the corners, so the arrangement of three points in this region is unique. Since we can fit at most three points in this region, we must put at least nine points in the lower region.

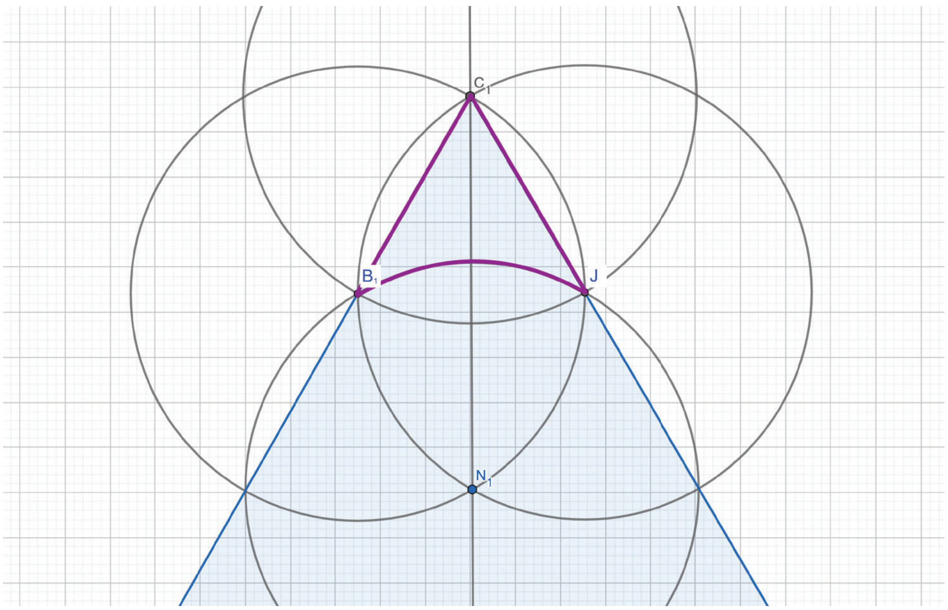

Figure 4. Equilateral triangle with upper region outlined

The decomposition shown in **Figure 5** partitions the lower region into eight subregions: the interiors of circles with radius d_{13} about U and Z (the highest points in the lower region), the interiors of the circles with radius d_{13} about the bottom corners, the interiors of the circles with radius d_{13} about V and T (the intersection points for the first two circles) minus the first two circles, and then the remaining space split symmetrically down the center. The lower region includes its upper boundary, so each of the five arcs that make up this boundary are included in their respective subregions. The purple and orange subregions include their boundaries except where they intersect with brown subregions; these intersections belong to the brown subregions. The brown subregions include their boundaries except where they intersect with the blue subregions. The blue subregions include their entire boundaries.

Since there are nine points, without loss of generality, we must place five left of and including the center. There are five points to place in four subregions. Thus, at least one subregion must have at least two points. Since the circular regions do not include their entire boundaries, they cannot fit more than one point. We take a closer look at the region outlined in blue from **Figure 5** in **Figure 6** below.

Figure 6 shows the region outlined in blue along with a circle of radius d_{13} about each of its corners. The circles about the corners of the region - N_2, H_1, G_1, and M_1 - are green, pink, orange, and brown respectively. Since the pink and brown circles contain the entire region, it is clear that any point in the blue region is less than d_{13} from H_1 and M_1. From Melissen's orginal paper, we know that N_2 (the point d_{13} from and directly below N_1) and G_1 (the point d_{13} from V on the side of the triangle) are d_{13} apart.[2] Thus, the only way to fit two points in this region is at N_2 and G_1.

We have established that the only subregion from **Figure 5** that can fit two points is the blue region and we cannot fit three points in any subregion. So we must put two points in the blue subregion at N_2 and G_1 and one point in each of the other subregions. We know from our lemma that there must be a point at A_3 (the corner). Now the only point

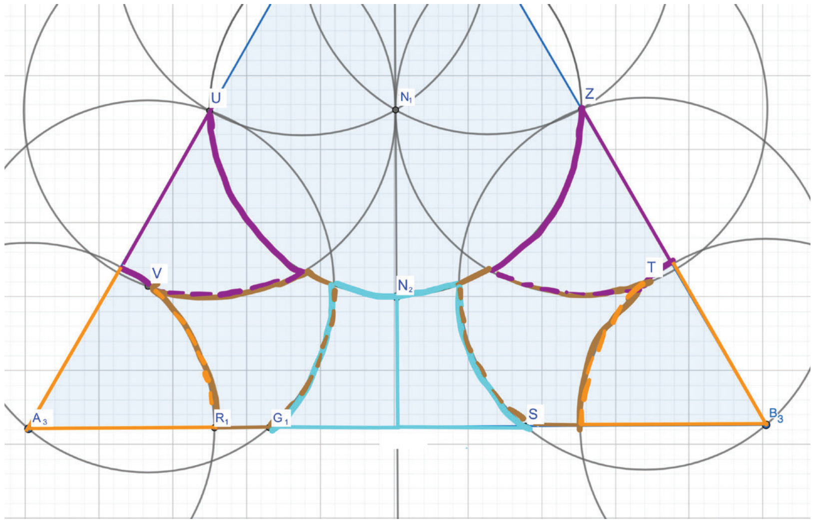

Figure 5. Decomposition of lower region of equilateral triangle

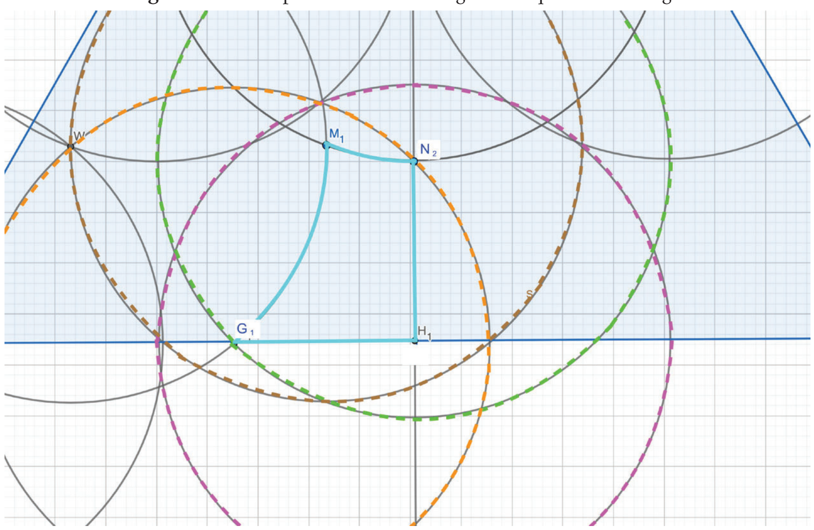

Figure 6. Blue subregion with circles about each corner

in the brown subregion that is d_{13} from G_1 and A_3 is V so we must place the next point there. The only point in the purple region that is d_{13} from V is U, so a point belongs there as well. By the same argument, the points on the right side must be placed in the corresponding locations. We have shown that we can fit at most three points in the upper region and nine points in the lower region and that the arrangements of points in both of those regions is unique, so we have uniquely placed all twelve points. This concludes **Theorem 1**.

Theorem 2. *Let N_2 from **Figure 2** be the point on the altitude from C_1 that is d_{13} from N_1. If N_2 is fixed, then Joos's Theorem for the optimal arrangement of 13 points is correct.*

In order to prove **Theorem 2**, we will first split the triangle into an upper and lower region and decompose the upper region to show that we can fit at most five points in the upper region. We will then decompose the lower region into identical halves, providing that one of these halves must contain four points. We will then show that an arrangement of four points into one of the halves is unique. Fixing these points will further limit the arrangements of points in the upper region, allowing us to change our decomposition and prove that we can actually only fit four points in the upper region, so therefore, there are four points in the other half of the lower region as well. We will then prove that there is a unique configuration of four points in the remaining half of the lower region and four points in the upper region, thus giving a unique configuration of all of the remaining twelve points.

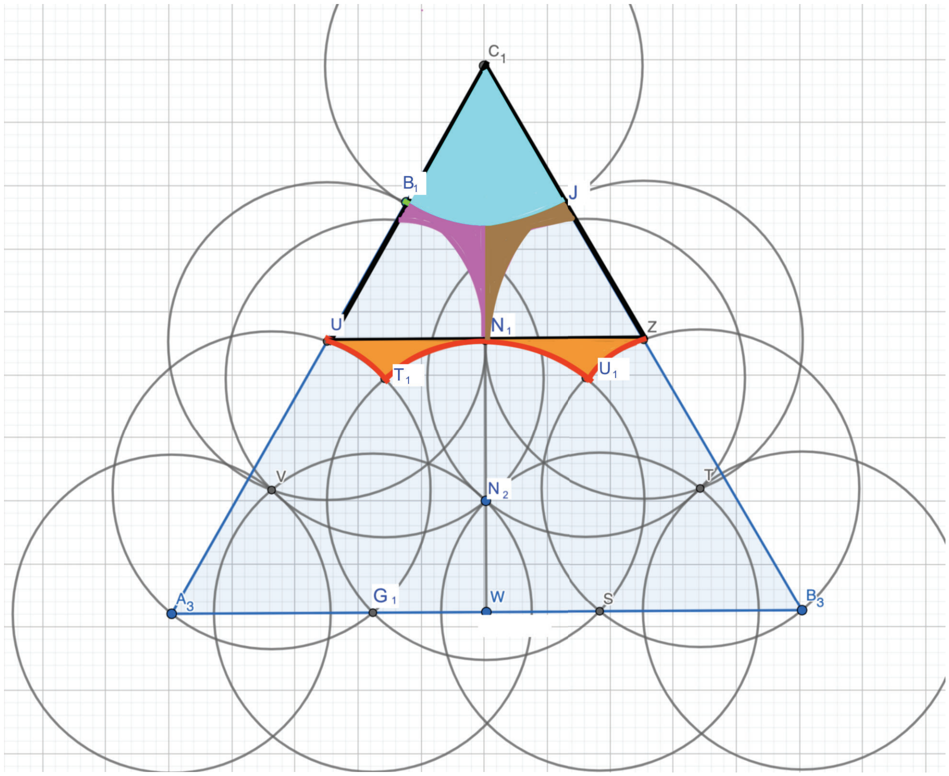

Figure 7. Decomposition of an upper region of an equilateral triangle

In order to maintain at least d_{13} between each point, there cannot be another point within a circle of radius d_{13} centered at N_2. **Figure 7** draws a red curve along the boundary of the union of a circles of radius d_{13} about the points N_2, V and T from **Figure 2**. We will consider the region above and including the red curve - but excluding the points U and Z - as the upper region, and the region below the red curve plus U and Z as the lower region.

Consider equilateral triangle $\triangle C_1UZ$ outlined in black in **Figure 7**. We will first show that there can be at most four points in this triangle excluding U and Z. Each side of this triangle has length $2d_{13}$ since the points are defined such that B_1 is d_{13} from C_1 and U is d_{13} from B_1. By Melissen's proof optimizing six points in an equilateral triangle, we can uniquely fit six points into this triangle such that no pair of points is less than d_{13} apart at C_1, B_1, J, U, Z, and N_1.[2] Since this configuration is unique, if we exclude points U and Z from the region, we can no longer fit six points in the triangle outlined in black. Furthermore, since the optimal configuration of five points in an equilateral triangle requires five points from the configuration of the six points labeled in the region,[2] removing those two points from the region makes it so that we cannot fit five points d_{13} apart from one another either. Thus, in the equilateral triangle outlined in black excluding U and Z, we can fit at most four points.

It follows that in order to fit six points in the upper region, we would need to put two points between the black triangle and the red curve shaded in orange - one on each side of N_1 since we cannot fit two points on either one side given that N_1 is d_{13} from U and Z. We will consider these two orange regions separated by N_1.

In order for there to be a point in each of these regions, there can be no point in the intersection of the circles about N_1, U, and T_1 and the intersection of circles about N_1, U_1, and Z all of radius d_{13}, i. e. there can be no points below the pink and brown regions in **Figure 7**. This is because any point that is in one of those intersections would be less than d_{13} from every point in the corresponding orange region, and we have already established that there must be a point in both regions for there to be six points in the whole upper region.

We next show that we cannot fit four points in the union of the blue, pink, and brown regions. The blue region is defined as the interior of a circle of radius d_{13} about C_1, so there cannot be more than one point in this region. The pink and brown regions can each also have at most one point. To understand why, keep in mind, we took $B_1, J,$ and N_1 from the optimal placement of six points in black equilateral triangle, so we know that they are d_{13} apart. We cannot have a point at N_1 because we know that there must be a point in each of the orange regions and U and Z are excluded from those regions, so each of those regions contains a point less than d_{13} from N_1. Thus, none of the blue, pink, and brown regions can fit more than one point, so we can fit at most three points the union of these three regions. Thus, there are at most five points in the upper region.

We now consider the lower region, which is below the red curve from **Figure 7** and includes U and Z, but excludes the interior of a circle of radius d_{13} about N_2. Since we know we can only fit five points in the upper region, we must fit seven points in this region. Without loss of generality, we must fit four point on the left side of this region.

Figure 8 decomposes the left side of this lower region into three subregions. We define the orange subregion as the interior of a circle of radius d_{13} about U, the green subregion as the interior of a circle of radius d_{13} about A_3 and the brown as the remaining space. We know from Melissen's paper that G_1 and V are d_{13} apart.[2] Thus, the orange and green subregions can each contain at most one point and the brown subregion can contain at most two points uniquely placed at G_1 and V. Since we need to fit four points into these three subregions, we need to fit at least two points in at least one of these subregions, so we must put points at G_1 and V. The only way to put a point in each of the remaining subregions is by putting them at A_3 and U. Thus, the only possible configuration of four points on the left side of this region are G_1, V, A_3, and U.

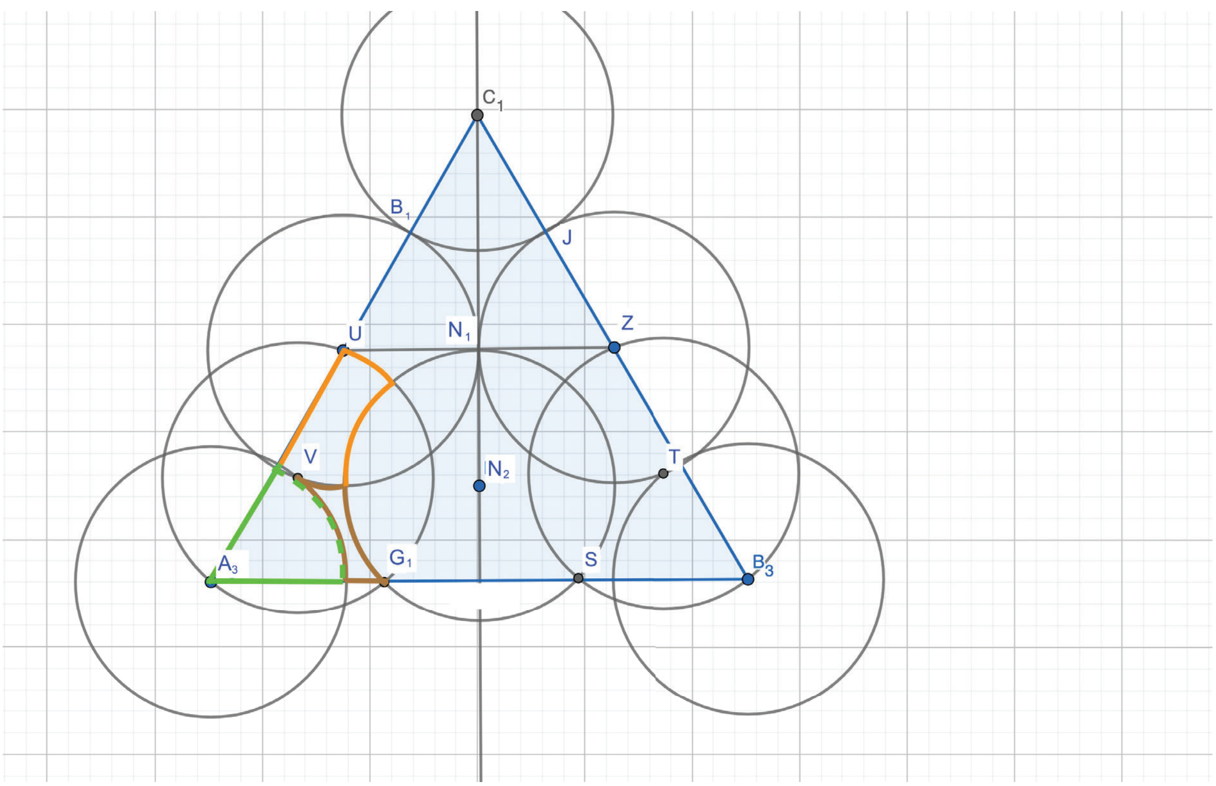

Figure 8. Decomposition of a left lower region of an equilateral triangle

Once we fix those four points, we cannot have any other points within d_{13} of any of them. In particular we cannot have any point in the interior of the circle of radius d_{13} about U which is outlined in pink in **Figure 9**. With this in mind, **Figure 9** revises our decomposition of the upper region using circles of radius d_{13} about C_1 and Z as boundaries of the regions. Note that Z is not in the upper region at all, N_1 is in the red region, J is in the yellow region, and B_1 is in the blue region. Since each region can contain at most one point, we can now conclude that there are at most four points in this entire upper region. This means that there must also be four points in the right side of the lower region. An analogous decomposition to that shown in **Figure 8** shows that the points on the right side of the lower region must be in analogous locations to those on the left: B_3, S, T, and Z. In particular, there is a point at Z, so the points in the red and yellow subregions in **Figure 9** must be at N_1 and J respectively. Fixing those points also requires that the only points that can be placed in each of the other two regions are B_1 and C_1. This concludes the unique configuration of the remaining twelve points.

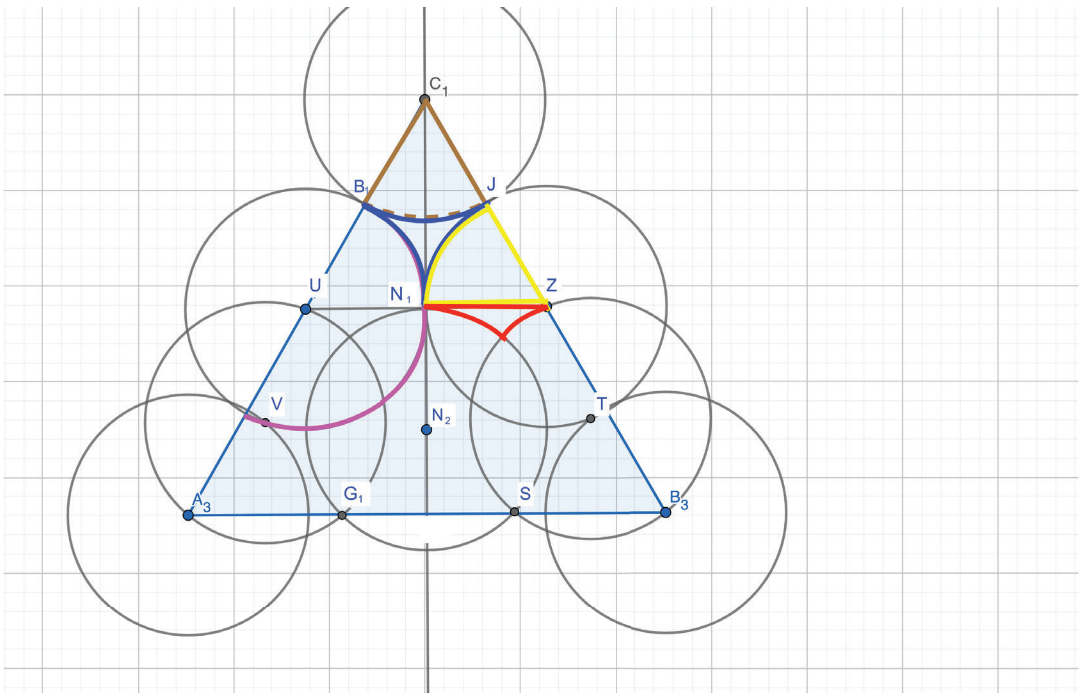

Figure 9. Refined decomposition of the upper region of the equilateral triangle

DISCUSSION

The proofs provided in this paper bring us closer to a discrete proof of Joos's Theorem for the optimal configuration of thirteen points in an equilateral triangle. We have shown that if either of two points is fixed, then we have an alternate proof of the Melissen configuration of thirteen points. We will now explore how one might begin proving the position of N_1 or N_2.

Theorem 3. *Recall N_1 and N_2 from **Figure 2**. Let K, T_1 and W_1, V_1 be in the position that N_1 and N_2 would be in if the entire triangle was rotated about the center such that C_1 was positioned at B_3 and A_3 respectively. Let region H be the orange hexagon from **Figure 10** defined by T_1, N_1, V_1, K, N_2, and W_1 including its boundary. If there are two points in this region at least d_{13} apart, they must be at either N_1 and N_2, K and T_1, or W_1 and V_1 (which by our definition are all the same pair of points up to rotations of the triangle).*

Since H is a convex hexagon and congruent across all three altitudes of the large triangle, proving **Theorem 3** is equivalent to proving that the distances from N_1 to W_1, N_1 to K, and W_1 to K are less than d_{13}.

The point E in **Figure 10** is the midpoint of the segment from C_1 to A_3 which is the side length of the triangle. Assuming the triangle has side length 1, the segment from C_1 to E is 0.5 in length. The segment between C_1 and U from **Figure 2**, on the other hand, has length $2d_{13} \approx 0.503626$. So E from **Figure 10** is closer to B_1 than U from **Figure 2** is. Therefore the distance between E and N_1 is less than d_{13} and equal to the distance between W_1 and E since W_1 is the same distance from A_3 as N_1 is from C_1. Thus, $\triangle EN_1W_1$ is isosceles, so $\angle EN_1W_1 \cong \angle EW_1N_1$. Since $\triangle C_1UZ$ from **Figure 2** is equilateral, $\angle C_1UN_1$ is 60°, so $\angle C_1EN_1$ in **Figure 10** is greater than 60°as is $\angle A_3EW_1$.

Thus, $\angle N_1EW_1$ is less than 60°, and therefore smaller than $\angle VN_1W_1$ and $\angle EW_1N_1$. So, the segment from W_1 to N_1 is less than that from E to N_1 which is less than d_{13}. We can generalize this argument to now say that the segments from N_1 to W_1, N_1 to K, and W_1 to K are less than d_{13} since they are all the same pair of points up to rotations of the triangle. This concludes **Theorem 3**.

Now in order to show that there are points at N_1 and N_2, we need only to show that there must be two points in H because the only way to fit two points in H is by placing points N_1 and N_2, K and T_1, or W_1 and V_1 which are all the same pair of points up to rotations of the triangle.

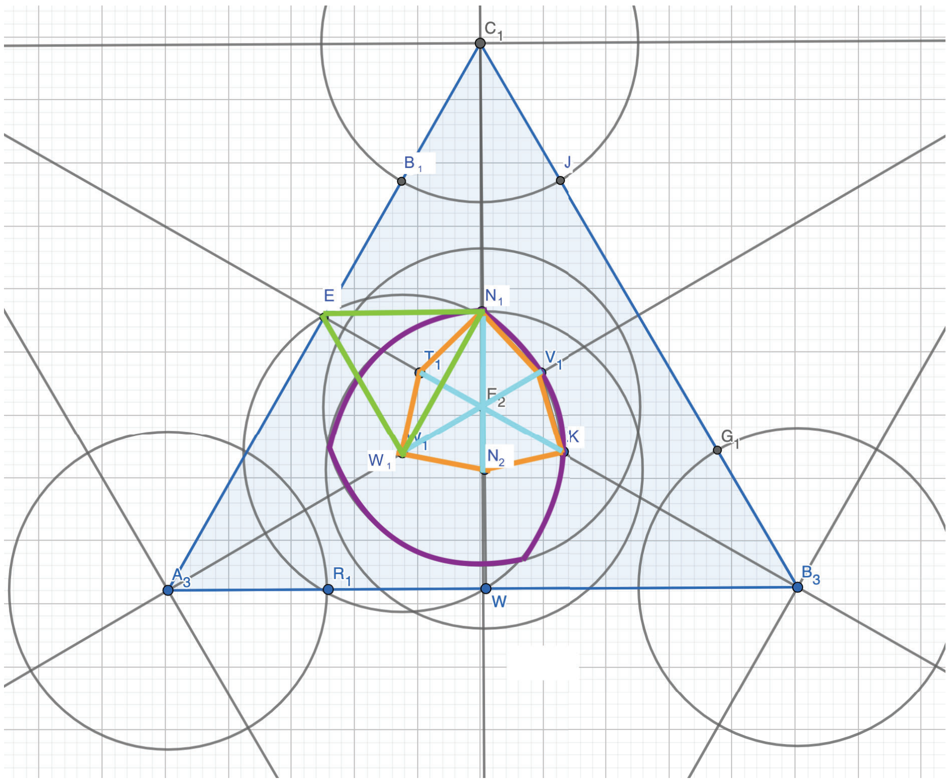

Figure 10. Equilateral triangle with regions of emphasis outlined

In order to show that there are at least two points in H, we would first need to show that there must be at least one point in H, meaning it is impossible to fit thirteen points outside of H that are at least d_{13} apart. We would then need to show that given that there is a point within H, we cannot fit twelve points outside of H that are all at least d_{13} apart. The blue lines clearly partition H into six subregions. Because the six subregions are congruent, once we prove that there is a point in H, we can assume without loss of generality that that point is in the bottom left subregion of H, the triangle defined by the points F_2, W_1, N_2 including the boundary.

If there is a point in that bottom left subregion of H, there cannot be another point that is less than d_{13} from all three corners of the subregion or else it would be less than d_{13} from every point in the subregion. Thus, if there is a point in the bottom left subregion of H, there cannot be any other point within the intersection of the circles of radius d_{13} about F_2, W_1 and N_2 which is outlined in purple (excluding the boundary); we will call this region P. So in order to complete the discrete proof of Joos's Theorem, all that remains to be proven is that we cannot place thirteen points outside of H and that we cannot fit twelve points outside of the union of H and P.

CONCLUSIONS

This paper makes progress towards a discrete proof for the optimal orientation of thirteen points in an equilateral triangle by proving that if either of two points is fixed, the other twelve are positioned in accordance with Joos's Theorem. Finding that such a proof exists could influence how mathematicians approach similar packing problems. It is also interesting to note that 13, along with 8 and 19 can be expressed as $\frac{k(k+1)}{2} - 2$ or $\frac{k(k+1)}{2} + 2k + 1$ depending on the value of k. The first representation shows us that these values are two less than triangular numbers (which are expressed as $\frac{k(k+1)}{2}$), but the second makes sense of the arrangement of the points. Melissen's proof for the optimal configuration of $n = 8$ points, and in his conjectures for $n = 13$ and $n = 19$, all have very similar arrangements. They all have an upper region where the points are arranged as an equilateral triangle with a triangular number of points (that corresponds with the first term of the expression). A discrete proof for $n = 13$ could make leeway for a general proof for all n that are two less than a triangular number.

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PRESS SUMMARY

How large is the smallest computer chip that contains n transistors that must be some distance apart from one another? How big must a crate be in order to hold n jugs of water? These questions ask how we can maximize the benefit of costly or harmful materials and provide insight on how we can drive technology and innovation forward. Mathematically, these and many more questions are the same. This paper furthers the discussion of solving these types of problems.