Strategies for Making Best Offers on eBay[a]

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ABSTRACT

We model for "Buy-It-Now or Best Offer" auctions on eBay using two different models. In the first model, risk-neutral bidders submit bids in serial and try to surpass a stochastic seller threshold while taking into account how many previous failed bids were made by other bidders. We compute optimal strategies for this model and show that bidder expected surplus decreases in the number of previous failed bids. In the second model we assume bidders do not know how many previous failed bids have been made, and instead use a first-price sealed-bid mechanism with a buy-out price where bidders serially submit bids with the knowledge that no previous bidders have used the buy-out price. We derive a unique equilibrium bidding strategy for risk-neutral bidders in this serial model, show that any equilibrium in a similar parallel bidding model is the same as the equilibrium in the serial model, and compute seller revenue. In particular, under certain circumstances, bidders will bid more in this format than they would in a standard first-price sealed-bid auction, but that a seller maximizes expected revenue by setting a buy-out price higher than any bidder is willing to pay thereby making the auction essentially a first-price auction.

KEYWORDS

Auction Theory; eBay; Buy-It-Now or Best Offer; Symmetric Bayesian Nash Equilibrium; Buy-Out Price; First-Price Sealed-Bid

DEDICATION

We dedicate this paper to our coauthor and friend Jamie Oliva (1994–2016).

INTRODUCTION

eBay is one of the internet's largest hubs of person-to-person commerce. The website hosts online auctions where sellers can sell goods directly to consumers and has been extensively studied by economists.[1] The standard auction on eBay, dubbed by some as a *California auction*,[2] is a timed auction where bidders may submit bids at any time. A bidder submits a maximum amount he or she[b] is willing to pay for the item, and any time another bidder bids an amount less that that maximum amount, eBay will bid on behalf of that original bidder the smallest possible bid needed to exceed the competing bid (eBay calls this system "proxy bidding"). In 2000 eBay also allowed sellers to include a "Buy-It-Now" price on these auctions, where any bidder could end the auction early by bidding the fixed "Buy-It-Now" price (abbreviated as BIN). This California auction mechanism is well-studied, and Reynolds and Wooders studied the effect of the inclusion of a "Buy-It-Now" price in that type of auction.[3] In this paper we consider another selling mechanism on eBay. eBay offers a format where a seller can list an item at a fixed "Buy-It-Now" price but has the option to consider offers from buyers for less than that price. eBay labels such an auction as a *Buy-It-Now auction with Best Offer option*, but we will refer to such auctions as simply *Best Offer auctions*. Instead of a base format of an ascending English auction as in the California auction, the Best Offer auction is a close cousin of a first-price sealed-bid auction. In such an auction, bidders secretly submit offers to the seller and the auction winner has to pay her offer. The Best Offer auction is different than a standard first-price sealed-bid auction in that there is a

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[b]In the future when a generic pronoun is called for, the authors will default to the feminine.

buy-out price (the BIN price) that a bidder can submit and win the item with certainty. For an example of such an auction, see **Figure 1**. The aim of this paper is to mathematically model optimal strategies for bidding in such Best Offer auctions.

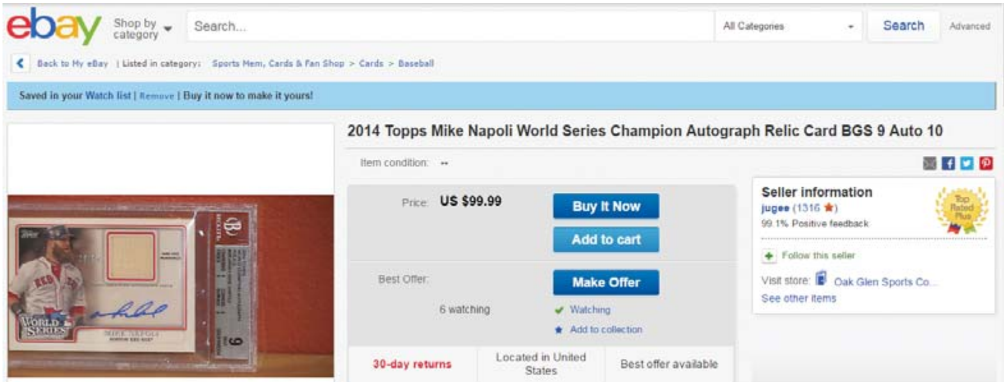

Figure 1. An example of a Best Offer auction page on eBay. Note the options to either "Buy It Now" or "Make an Offer."

When deciding to place a bid (either an offer or the BIN), a bidder can take into consideration other pieces of information that could influence her bid. In the past, eBay would include the number of previous bids made (but not the amount of those bids). This information included whether or not the offer was "pending" (that is, still under consideration by the seller) or "declined." In our first model of the Best Offer auction, which we will call the *Secret Threshold game* or just *ST game*, we assume every bidder knows how many previous bids the seller has declined. We exclude the number of pending bids to simplify the model, and to isolate the impact of the number of previous failed bids on optimal bidding behavior. The investigation of this model is the subject of the first section of the paper.

Several years ago, eBay changed the type of information available to prospective bidders. No longer is the number of previous offers (both pending or declined) public. Instead, now the number of "watchers" is public. On eBay, users can "watch" an auction of any format. This means that the user selects the auction to be saved as a part of that user's personal "watch list." However, it is not entirely clear how much useful knowledge a bidder can glean from the number of users watching a Best Offer auction, as watchers need not be past nor future bidders. To account for this uncertainty, we will model the Best Offer auction as a first-price sealed-bid auction with a buy-out price where the number of bidders is unknown but satisfies some given probability distribution. The unknown bidder structure we use is akin to the model developed by McAfee and McMillan.[4]

Generally, the strategies for a first-price sealed-bid auction with an unknown number of bidders is known,[4,5] however our main model for this scenario, which we dub the *Serial First Price Buy-Out (SFPBO) auction*, has two other wrinkles. The first wrinkle, as previously stated, is that there is a buy-out price, and thus there should be a threshold value over which a bidder will decide to use the buy-out price instead of submitting a bid. The second wrinkle is that we assume that bidders will place bids in serial in a randomly prescribed but unknown order. Generally in a first-price sealed-bid auction, bidders are modeled submitting bids simultaneously. However, in practice on eBay, bidders will see and bid on an Best Offer auction at different times. The mere fact that a Best Offer auction has not yet ended gives a bidder some information. Specifically, the bidder in this case knows that any previous bidders have submitted offers lower than the BIN. Our model will account for this serial nature of bid submission and this extra information available to bidders.

In the second section of the paper, we derive an equilibrium strategy for risk-neutral bidders for the SFPBO auction. In the third section we compare this equilibrium to an equilibrium to a more standard first-price with buy-out model which we will dub the *First Price Buy-Out (FPBO) auction* where bids are submitted simultaneously, and show that any equilibrium for the FPBO auction must be the same as the equilibrium for the SFPBO auction. Finally, in the final section, we will see that, in equilibrium a seller maximizes revenue by setting the buy-out price higher than any risk-neutral bidder is willing to pay. Thus, a seller in this setting does best by making the auction effectively a first-price sealed-bid auction without a reserve price.

THE SECRET THRESHOLD GAME

We start by considering a model, which we will call the *Secret Threshold (ST) game*, where bidders will know how many previous failed offers have been placed on the item up for bid. One important simplification we are making here (and as well in our next model) is that we are assuming each bidder can submit only one offer for the item. On eBay, buyers can submit up to three offers, and sellers have the option to make counteroffers on the first two offers. We do not include this possibility in our model as we are not interested in studying negotiation strategy, but the effect of the knowledge of previously failed bids on optimal bid making. Note that in any event, if the buyer makes three offers, the last offer is necessarily a final take-it-or-leave-it offer, which matches our model. On eBay, even if the seller declines an offer, the buyer can still purchase the item through the Buy-It-Now option. In our model we do not include such an option, as we are more interested in how a bidder crafts a single optimal offer free of negotiation. However, our model below could be modified to allow for this option.

We call this model an ST game because in this model bidders are trying to exceed (by as thin of as margin possible) an unknown stochastic threshold set by a seller. While we use the generic term of "threshold" here, in auction theoretic terms this threshold is just the seller's reservation price. We call this a "game" rather than an "auction" since bidders do not directly compete against one another in this model. Again, this is a simplification as eBay gives sellers 48 hours to respond to an offer, and so certainly within that time frame a seller could have multiple "pending" offers. However, in this model, we are less concerned with the effects of competition among the bidders and more on the effects of bidders knowing the number of failed bids. With that said, below we give the mathematical assumptions of the ST game.

Definition 1. *The conditions of Secret Threshold (ST) game are:*

(i) *There are a countable number of bidders. For $n \in \mathbb{N}$, the n^{th} bidder's value is the random variable V_n with range $[0, 1]$ and has a cumulative distribution function (cdf) $F(v)$ and a probability density function (pdf) $f(v)$. Note that this means all bidder values are identically distributed.*

(ii) *The i^{th} bidder knows her own value v_i for the item and how many failed bids have been previously made. That is, the first bidder knows she is the first bidder and if $n > 1$, the n^{th} bidder knows there have been $n-1$ previous failed bids. The n^{th} bidder does not know the values of V_i for $i \neq n$, but does know they have followed the distribution $F(v)$.*

(iii) *There is a value $0 < \omega \leq 1$, called the* buy-out price, *at which the seller is guaranteed to sell the item. Furthermore, there is a random variable B_0 with range of $[0, \omega]$ with the cdf $H(b)$ and pdf $h(b)$. The seller will accept any offer that exceeds B_0 (that is, B_0 represents the "secret threshold" of this game). That is, if the n^{th} bidder submits the bid b, if $b \geq B_0$ then this bidder wins the auction and gains the surplus $v_n - b$. If not, the bidder does not win and has a surplus of 0. That is, the* surplus function, *Π^{ST}, is given by*

$$\Pi^{ST}(v_n, b) = (v_n - b)\mathbb{1}_{b \geq B_0}, \qquad \text{Equation 1.}$$

where $\mathbb{1}_{b \geq B_0}$ is the indicator random variable which equals 1 when $b \geq B_0$ and equals 0 when $b < B_0$.

(iv) *The random variables $B_0, V_1, V_2, V_3, \ldots$ are all independent.*

(v) *The pdf $h(b)$ is positive and continuous on $[0, \omega]$ and the pdf $f(v)$ is positive and continuous on $(0, 1)$.*

The last two assumptions above are technical mathematical impositions to simplify the model. In particular, one could assume some positive correlation between the bidder values $V_1, V_2, \ldots$ (often in auction theory one assumes that values are *affiliated*[2, 5, 6]) but we will keep things simpler here. Our goal here is to find a bid that will maximize the expected surplus for the n^{th} bidder given that they know there have been $n-1$ failed bids before them. That is, for bidder 1, we want a bid $b_1(v_1)$ that will maximize $E[\Pi^{ST}]$. For any bidder $n > 1$, we want to maximize the conditional expected value

$E[\Pi^{ST} | \max_{1\leq i\leq n-1} b_i(V_i) < B_0]$. That is, the n^{th} bidder assumes the previous bidders bid optimally and still lost, and thus conditions her expectation on that fact. As is suggested by the previous discussion, we will inductively create these bidding functions $b_n : [0,1] \to [0,\omega]$.

The First Bidder

First we will consider the first bidder in a ST game. Our bidder would like to find a bid b_1 that maximizes her expected surplus given that her value is v_1. Note that by assumption (iii) of the ST game, we have

$$E[\Pi^{ST}(v_1, b_1)] = (v_1 - b_1)P(b_1 \geq B_0) = (v_1 - b_1)H(b_1). \qquad \textbf{Equation 2.}$$

In an attempt to maximize the above, we take the derivative with respect to b_1 and set the resulting quantity to 0 to look for critical points:

$$0 = \frac{d}{db_1}\Big(E\big[\Pi^{ST}\big]\Big) = -H(b_1) + (v_1 - b_1)h(b_1). \qquad \textbf{Equation 3.}$$

Solving the above for v_1 gives us

$$v_1 = b_1 + \frac{H(b_1)}{h(b_1)}. \qquad \textbf{Equation 4.}$$

The above equation motivates the following definition.

Definition 2. *Assume the conditions of the ST game. Define a function* $\phi_1 : [0,\omega] \to [0,\infty)$ *as*

$$\phi_1(x) = x + \frac{H(x)}{h(x)}. \qquad \textbf{Equation 5.}$$

The function ϕ_1 is very similar but not equal to to the marginal revenue function $\mathrm{MR}(x) = x - \frac{1-H(x)}{h(x)}$ often cited in the Revenue Equivalence Theorem. Further, we note that ϕ_1 is continuous.

Proposition 3. *The function* ϕ_1 *defined in* ***Definition 2*** *is continuous on* $[0,1]$.

Proof. Note that since h is continuous on $[0,1]$ and positive by assumption (v), then ϕ_1 is also continuous on $[0,1]$. □

We can now state our first theorem.

Theorem 4. *Assume the conditions of the ST game and suppose the function* $\phi_1(b) = b + \frac{H(b)}{h(b)}$ *is increasing on* $[0,\omega]$. *The bidding function* $b_1 : [0,1] \to [0,\omega]$ *that uniquely maximizes the expected surplus* $E[\Pi^{ST}]$ *is*

$$b_1(v) = \begin{cases} \phi_1^{-1}(v) & \textit{if } v \leq \phi_1(\omega) \\ \omega & \textit{else} \end{cases}. \qquad \textbf{Equation 6.}$$

Note that the above bidding strategy has a nice form. It tells us exactly when to use the BIN ω instead of sending an offer (when $v \leq \phi_1(\omega)$). Also, in the above theorem we assume that ϕ_1 is increasing. This is akin to the simplifying (but not necessary) assumption found in Myerson that the marginal revenue function MR is increasing when finding an optimal auction mechanism.[7]

Proof. By the previous proposition, ϕ_1 is continuous. As ϕ_1 is also assumed to be increasing on its domain, it is invertible and we have $\phi_1^{-1} : [\phi_1(0), \phi(\omega)] \to [0,\omega]$. In fact, evaluating ϕ_1 at 0 yields $\phi_1(0) = 0 + 0 = 0$. So, we have $\phi_1^{-1} : [0, \phi(\omega)] \to [0,\omega]$. We have two cases to consider: the case where $v_1 \leq \phi(\omega)$ and the case where $v_1 > \phi(\omega)$. First, suppose $v_1 \leq \phi_1(\omega)$ so that $\phi_1^{-1}(v_1)$ is defined. If $b_1 < \phi_1^{-1}(v_1)$, then $\phi_1(b_1) < v_1$ and thus,

$$\frac{d}{db_1}\Big(E\big[\Pi^{ST}\big]\Big) = -H(b_1) + (v_1 - b_1)h(b_1) > -H(b_1) + \big(\phi_1(b_1) - b_1\big)h(b_1) = 0. \qquad \textbf{Equation 7.}$$

This shows that $E\left[\Pi^{ST}\right]$ is increasing for $b_1 < \phi_1^{-1}(v_1)$. On the other hand, if $b_1 > \phi_1^{-1}(v_1)$, then $\phi_1(b_1) > v$ and thus

$$\frac{d}{db_1}\left(E\left[\Pi^{ST}\right]\right) = -H(b_1) + (v_1 - b_1)h(b_1) < -H(b_1) + \big(\phi_1(b_1) - b_1\big)h(b_1) = 0, \qquad \textbf{Equation 8.}$$

showing that $E\left[\Pi^{ST}\right]$ is decreasing for $b_1 < \phi_1^{-1}(v_1)$. Thus, $b_1 = \phi_1^{-1}(v_1)$ is the bid that maximizes the expected surplus $E\left[\Pi^{ST}\right]$ when $v_1 \leq \phi_1(\omega)$.

Next, suppose $v_1 > \phi_1(\omega)$. Then we have $v_1 > \phi_1(\omega) \geq \phi_1(b_1)$ for any $b_1 \leq \omega$. So

$$\frac{d}{db_1}\left(E\left[\Pi^{ST}\right]\right) = -H(b_1) + (v_1 - b_1)h(b_1) > -H(b_1) + (\phi_1(b_1) - b_1)h(b_1) = 0. \qquad \textbf{Equation 9.}$$

So $E\left[\Pi^{ST}\right]$ is increasing for all bids b_1. Thus, the quantity is maximized at the maximum bid ω. □

For an example, suppose the secret reserve B_0 has a uniform distribution on $[0, \omega]$. Then $H(b) = \frac{1}{\omega}b$ and $h(b) = \frac{1}{\omega}$, giving us $\phi_1(b) = 2b$ and thus $\phi_1^{-1}(v) = \frac{1}{2}v$. Thus,

$$b_1(v) = \begin{cases} \frac{1}{2}v & \text{if } v \leq 2\omega \\ \omega & \text{else.} \end{cases} \qquad \textbf{Equation 10.}$$

Bidding After Failed Bids

Now let's consider the strategy for the n^{th} bidder where $n > 1$. The n^{th} bidder is bidding after $n-1$ failed bids and thus wants to maximize the conditional expected value

$$E\left[\Pi^{ST}(v_n, b)\,\middle|\, \max_{1 \leq i \leq n-1} b_i(V_i) \leq B_0\right] \qquad \textbf{Equation 11.}$$

in b. In the theorem below, we establish an inductive formula for constructing an optimal strategy for the n^{th} bidder.

Theorem 5. *Let $n > 1$ and assume the conditions of the ST game. Suppose for $1 \leq i \leq n-1$ there are bidding functions b_i defined for the i^{th} bidder in a ST game of the form*

$$b_i(v) = \begin{cases} \phi_i^{-1}(v) & \textit{if } v \leq \phi_i(\omega) \\ \omega & \textit{else} \end{cases}, \qquad \textbf{Equation 12.}$$

where ϕ_i is continuous and increasing on $[0, \omega]$ and $\phi_i(0) = 0$. Define the function $\phi_n(b)$ as

$$\phi_n(b) := \begin{cases} b + \dfrac{\int_0^b \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0}{\left(\prod_{i=1}^{n-1} F(\phi_i(b))\right) h(b)} & \textit{if } b > 0 \\ 0 & \textit{if } b = 0 \end{cases}. \qquad \textbf{Equation 13.}$$

Then $\phi_n(b)$ is continuous. Furthermore, if $\phi_n(b)$ is increasing, the function $b_n(v)$ defined as

$$b_n(v) = \begin{cases} \phi_n^{-1}(v) & \textit{if } v \leq \phi_n(\omega) \\ \omega & \textit{else} \end{cases} \qquad \textbf{Equation 14.}$$

uniquely maximizes the expected surplus $E\left[\Pi^{ST} \,\middle|\, \max_{1 \leq i \leq n-1} b_i(V_i) \leq B_0\right]$ of the n^{th} bidder in a ST game.

Proof. First, note that the function ϕ_n defined in **Equation 13** is well-defined for $b > 0$ by the fact that each ϕ_i is increasing, $\phi_i(0) = 0$, and by assumption (v) of the ST game. Since each ϕ_i is continuous and h and F are continuous by assumption (v), ϕ_n is continuous on $(0,\omega]$. To check continuity at $b=0$, note that since $F(\phi_i(b))$ is increasing, we have

$$\begin{aligned} 0 \quad \le \quad & \phi_n(b) = b + \frac{\int_0^b \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0}{\left(\prod_{i=1}^{n-1} F(\phi_i(b))\right) h(b)} \\ < \quad & b + \frac{\int_0^b \left(\prod_{i=1}^{n-1} F(\phi_i(b))\right) h(b_0)\, db_0}{\left(\prod_{i=1}^{n-1} F(\phi_i(b))\right) h(b)} = b + \phi_1(b), \end{aligned}$$

Equation 15.

showing (by the Squeeze Theorem) that $\phi_n(b) \to 0$ as $b \to 0^+$. Thus, ϕ_n is continuous on its domain.

To prove the second part of the theorem, we assume that ϕ_n is also increasing. For any bid b_n, we have

$$E\left[\Pi^{ST}(v_n, b_n)\,\middle|\, \max_{1\le i\le n-1} b_i(V_i) \le B_0\right] = (v_n - b_n) P\left(B_0 \le b_n \,|\, b_1(V_1) \le B_0,\, \ldots,\, b_{n-1}(V_{n-1}) \le B_0\right).$$

Equation 16.

Note that if $b_i(V_i) < B_0$, we know that bidder i did not bid the BIN, ω, meaning she must have bid $\phi_i^{-1}(V_i)$. By assumptions (i) and (iv) the values V_i are iid with cdf $F(v)$ and pdf $f(v)$ and the random variables $B_0, V_1, \ldots, V_{n-1}$ are all independent. Thus we have

$$\begin{aligned} P(b_1(V_1) \le B_0,\, \ldots,\, b_{n-1}(V_{n-1}) \le B_0) = \; & P(V_1 \le \phi_1(B_0),\, \ldots,\, V_{n-1} \le \phi_{n-1}(B_0)) \\ = \; & \int_0^\omega \int_0^{\phi_1(b_0)} \cdots \int_0^{\phi_{n-1}(b_0)} h(b_0) f(v_1) \ldots f(v_{n-1})\, dv_{n-1} \ldots dv_1 db_0 \\ = \; & \int_0^\omega \left(\prod_{i=1}^{n-1} \int_0^{\phi_i(b_0)} f(v_i)\, dv_i\right) h(b_0)\, db_0 = \int_0^\omega \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0. \end{aligned}$$

Equation 17.

Similarly, we can compute

$$P(B_0 \le b,\, b_1(V_1) \le B_0,\, \ldots,\, b_{n-1}(V_{n-1}) \le B_0) = \int_0^b \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0.$$

Equation 18.

Inserting **Equation 17** and **Equation 18** into **Equation 16** yields

$$E\left[\Pi^{ST}(v_n, b)\,\middle|\, \max_{1\le i\le n-1} b_i(V_i) \le B_0\right] = (v_n - b) \frac{\int_0^b \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0}{\int_0^\omega \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0}.$$

Equation 19.

Thus we can compute the derivative of $E[\Pi^{ST}(v_n, b) | \max_{1\le i\le n-1} b_i(V_i) \le B_0]$:

$$\frac{d}{db} E\left[\Pi^{ST}(v_n, b)\,\middle|\, \max_{1\le i\le n-1} b_i(V_i) \le B_0\right] = \frac{-\int_0^b \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0 + (v - b)\left(\prod_{i=1}^{n-1} F(\phi_i(b))\right) h(b)}{\int_0^\omega \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0}.$$

Equation 20.

Suppose that $v_n \le \phi_n(\omega)$. Then v_n is in the range of ϕ_n so we can write $\phi_n^{-1}(v_n)$. Next suppose $b < \phi_n^{-1}(v_n)$. Since ϕ_n is increasing, we have $\phi_n(b) < v_n$, so

$$\begin{aligned} \frac{d}{db} E\left[\Pi^{ST}(v_n, b)\,\middle|\, \max_{1\le i\le n-1} b_i(V_i) \le B_0\right] = \; & \frac{-\int_0^b \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0 + (v_n - b)\left(\prod_{i=1}^{n-1} F(\phi_i(b))\right) h(b)}{\int_0^\omega \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0} \\ > \; & \frac{-\int_0^b \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0 + (\phi_n(b) - b)\left(\prod_{i=1}^{n-1} F(\phi_i(b))\right) h(b)}{\int_0^\omega \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0} \\ = \; & 0, \end{aligned}$$

Equation 21.

so that $E\left[\Pi^{ST}(v_n,b)\,|\max_{1\le i\le n-1} b_i(V_i)\le B_0\right]$ is increasing for $b<\phi_n^{-1}(v_n)$. Now suppose $b>\phi_n^{-1}(v_n)$. Since ϕ_n is increasing, we have $\phi_n(b)>v_n$, so

$$\begin{aligned}\frac{d}{db}E\left[\Pi^{ST}(v_n,b)|\max_{1\le i\le n-1} b_i(V_i)\le B_0\right] &= \frac{-\int_0^b\left(\prod_{i=1}^{n-1}F(\phi_i(b_0))\right)h(b_0)\,db_0+(v_n-b)\left(\prod_{i=1}^{n-1}F(\phi_i(b))\right)h(b)}{\int_0^\omega\left(\prod_{i=1}^{n-1}F(\phi_i(b_0))\right)h(b_0)\,db_0}\\ &< \frac{-\int_0^b\left(\prod_{i=1}^{n-1}F(\phi_i(b_0))\right)h(b_0)\,db_0+(\phi_n(b)-b)\left(\prod_{i=1}^{n-1}F(\phi_i(b))\right)h(b)}{\int_0^\omega\left(\prod_{i=1}^{n-1}F(\phi_i(b_0))\right)h(b_0)\,db_0}\\ &=0,\end{aligned}$$ **Equation 22.**

so that $E\left[\Pi^{ST}(v_n,b)\,|\max_{1\le i\le n-1} b_i(V_i)\le B_0\right]$ is decreasing for $b<\phi_n^{-1}(v_n)$. Thus, $E\left[\Pi^{ST}(v_n,b)\right]$ is maximized at $b=\phi_n^{-1}(v_n)$ if $v_n\le\phi_n(\omega)$.

Next we consider the case where $v_n>\phi_n(\omega)$. In this case, for any $b<\omega$ we have $\phi_n(b)<\phi_n(\omega)<v_n$, so

$$\begin{aligned}&\frac{d}{db}E\left[\Pi^{ST}(v_n,b)|\max_{1\le i\le n-1} b_i(V_i)\le B_0\right]\\ &= \frac{-\int_0^b\left(\prod_{i=1}^{n-1}F(\phi_i(b_0))\right)h(b_0)\,db_0+(v_n-b)\left(\prod_{i=1}^{n-1}F(\phi_i(b))\right)h(b)}{\int_0^\omega\left(\prod_{i=1}^{n-1}F(\phi_i(b_0))\right)h(b_0)\,db_0}\\ &> \frac{-\int_0^b\left(\prod_{i=1}^{n-1}F(\phi_i(b_0))\right)h(b_0)\,db_0+(\phi_n(b)-b)\left(\prod_{i=1}^{n-1}F(\phi_i(b))\right)h(b)}{\int_0^\omega\left(\prod_{i=1}^{n-1}F(\phi_i(b_0))\right)h(b_0)\,db_0}=0,\end{aligned}$$ **Equation 23.**

showing that $E[\Pi^{ST}(v_n,b)|\max_{1\le i\le n-1} b_i(V_i)\le B_0]$ is always increasing, and thus is maximized on the right endpoint ω, proving the theorem. □

The above theorem gives an inductive way to find the bidding functions of each bidder. Note that in the statement of Theorem 5, we actually do not assert that for $1\le i\le n-1$ the strategy $b_i(v_i)$ maximizes the conditional expected surplus for bidder i, but only that b_i satisfies **Equation 12**. However, assuming rational bidders, the n^{th} bidder could assume all previous bidders bid optimally, so when we construct examples of these functions below, we will assume all bidders have used the optimal bidding function.

For an example, let's suppose the seller's reserve B_0 has a uniform distribution on $[0,\omega]$ and the other bidders distributions are also uniform over $[0,1]$. We will inductively show that the maximizing bidding function is

$$b_n(v)=\begin{cases}\left(1-\frac{1}{n+1}\right)(v) & \text{if } v\le\frac{n+1}{n}\omega\\ \omega & \text{else.}\end{cases}$$ **Equation 24.**

Note that in the base case, we have previously shown that

$$b_1(v)=\begin{cases}\left(1-\frac{1}{2}\right)v & \text{if } v\le 2\omega\\ \omega & \text{else,}\end{cases}$$ **Equation 25.**

which fits our formula when $n=1$. For the inductive step, let $n>1$ and assume the formula holds for all i such that $1\le i\le n-1$. Thus, for $1\le i\le n-1$

$$b_i(v)=\begin{cases}\left(1-\frac{1}{i+1}\right)(v) & \text{if } v\le\frac{i+1}{i}(\omega)\\ \omega & \text{else}\end{cases}$$ **Equation 26.**

Note that by the above formula, and by **Theorem 5** we have $\phi_i(b) = \frac{i+1}{i}b$. Also by **Theorem 5** we have

$$\begin{aligned}
\phi_n(b) &= b + \frac{\int_0^b \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0}{\left(\prod_{i=1}^{n-1} F(\phi_i(b))\right) h(b)} \\
&= b + \frac{\int_0^b \left(\prod_{i=1}^{n-1} F(\frac{i+1}{i}(b_0))\right) \frac{1}{\omega}\, db_0}{\left(\prod_{i=1}^{n-1} F(\frac{i+1}{i}(b))\right) \frac{1}{\omega}} \\
&= b + \frac{\int_0^b \left(\prod_{i=1}^{n-1} \frac{i+1}{i}(b_0)\right)\, db_0}{\left(\prod_{i=1}^{n-1} \frac{i+1}{i}(b)\right)} \\
&= b + \frac{\int_0^b \left(b_0^{n-1}\right)\, db_0}{(b^{n-1})} \\
&= b + \frac{\frac{b^n}{n}}{(b^{n-1})} \\
&= b + \frac{b}{n} \\
&= \frac{n+1}{n}b.
\end{aligned}$$

Equation 27.

Thus we have $\phi_n^{-1}(v) = \frac{n}{n+1}v = \left(1 - \frac{1}{n+1}\right)v$, and thus

$$b_n(v) = \begin{cases} \left(1 - \frac{1}{n+1}\right)(v) & \text{if } v \le \frac{n+1}{n}(\omega) \\ \omega & \text{else} \end{cases},$$

Equation 28.

completing the induction.

Ordering of The Bidding Functions

One might deduce from the uniform example above that bidding functions $b_n(v)$ generally increase as n increases. Generally one would suspect that bidders failing to win the item is a signal to subsequent bidders that the secret reserve is high and such bidders should be more aggressive in their bids. This is indeed the case.

Theorem 6. *Let $n > 1$. Suppose the conditions of the ST game and of* ***Theorem 5*** *are satisfied and the functions b_{n-1} and b_n are defined as in* ***Theorem 5****. If $v = 0$ or $v \ge \phi_{n-1}(\omega)$ we have*

$$b_{n-1}(v) = b_n(v).$$

Equation 29.

If $0 < v < \phi_{n-1}(\omega)$, then

$$b_{n-1}(v) < b_n(v).$$

Equation 30.

Proof. Note that $b_{n-1}(0) = 0 = b_n(0)$. Note that for $b > 0$, if we set the product $\prod_{i=1}^{0} F(\phi_i(b))$ to equal 1 to cover the case where $n = 2$, we have

$$\begin{aligned}
\phi_{n-1}(b) &= b + \frac{\int_0^b \left(\prod_{i=1}^{n-2} F(\phi_i(b_0))\right) h(b_0)\, db_0}{\left(\prod_{i=1}^{n-2} F(\phi_i(b))\right) h(b)} \\
&= b + \frac{\int_0^b \left(\prod_{i=1}^{n-2} F(\phi_i(b_0))\right) F(\phi_{n-1}(b)) h(b_0)\, db_0}{\left(\prod_{i=1}^{n-1} F(\phi_i(b))\right) h(b)} \\
&> b + \frac{\int_0^b \left(\prod_{i=1}^{n-1} F(\phi_i(b_0))\right) h(b_0)\, db_0}{\left(\prod_{i=1}^{n-1} F(\phi_i(b))\right) h(b)} = \phi_n(b),
\end{aligned}$$

Equation 31.

where the inequality is justified by the fact that $F(\phi_{n-1}(b_0))$ is an increasing function. We will repeatedly use the fact $\phi_{n-1}(b) > \phi_n(b)$ for $b > 0$ in what follows.

Suppose that $v \geq \phi_{n-1}(\omega)$. Then by the above we also have $v > \phi_n(\omega)$. Thus, we have

$$b_{n-1}(v) = \omega = b_n(v). \quad \text{Equation 32.}$$

We next will show that for all other values of v (i.e. $0 < v < \phi_{n-1}(\omega)$) we have the strict inequality $b_{n-1}(v) < b_n(v)$. Suppose that $\phi_n(\omega) \leq v < \phi_{n-1}(\omega)$. Note that $b_n(v) = \omega$. Also, since ϕ_{n-1} is increasing we have

$$b_{n-1}(v) = \phi_{n-1}^{-1}(v) < \phi_{n-1}^{-1}(\phi_{n-1}(\omega)) = \omega = b_n(v). \quad \text{Equation 33.}$$

Lastly, suppose $0 < v < \phi_n(\omega)$. Then $b_{n-1}(v) = \phi_{n-1}^{-1}(v) < \omega$ and $b_n(v) = \phi_n^{-1}(v) < \omega$. First note that

$$v = \phi_n(\phi_n^{-1}(v)) < \phi_{n-1}(\phi_n^{-1}(v)). \quad \text{Equation 34.}$$

Using the above, we have

$$b_{n-1}(v) = \phi_{n-1}^{-1}(v) < \phi_{n-1}^{-1}(\phi_{n-1}(\phi_n^{-1}(v))) = \phi_n^{-1}(v) = b_n(v), \quad \text{Equation 35.}$$

completing the proof. □

This increase of bids affects the conditional expected surplus. This surplus can also be affected by the conditional probability in the $n-1^{th}$ and n^{th} cases. In particular, one can show that for any fixed bid $b \leq \omega$, we have

$$P(B_0 \leq b \,|\, b_1(V_1) \leq B_0, \ldots, b_{n-2}(V_{n-2}) \leq B_0, b_{n-1}(V_{n-1}) \leq B_0) \leq P(B_0 \leq b \,|\, b_1(V_1) \leq B_0, \ldots, b_{n-2}(V_{n-2}) \leq B_0). \quad \text{Equation 36.}$$

Heuristically, the above equation is justified since $b_{n-1}(V_{n-1}) \leq B_0$ makes it no more likely that $B_0 \leq b$. It is a well-known fact that can be extended to random variables that are affiliated.[6] For the sake of self-containment, however, we will prove the above equation in the next lemma.

Lemma 7. *Let $k \in \mathbb{N}$ and suppose $X, Y_1, \ldots, Y_{k-1}, Y_k$ are independent random variables and d is any real number. Then*

$$P(X \leq d \,|\, Y_1 \leq X) \leq P(X \leq d), \quad \text{Equation 37.}$$

and if $k > 1$,

$$P(X \leq d \,|\, Y_1 \leq X, \ldots, Y_{k-1} \leq X, Y_k \leq X) \leq P(X \leq d \,|\, Y_1 \leq X, \ldots, Y_{k-1} \leq X). \quad \text{Equation 38.}$$

Proof. Suppose $X, Y_1, \ldots, Y_{k-1}, Y_k$ are independent. Note that for any nonnegative real numbers a, b, c such that $b > 0$ and $a < b$, we have

$$\frac{a}{b} \leq \frac{a+c}{b+c}. \quad \text{Equation 39.}$$

If $k > 1$, we have

$$\begin{aligned}
&P(X \leq d \,|\, Y_1 \leq X, \ldots, Y_{k-1} \leq X, Y_k \leq X) \\
&= \frac{P(X \leq d, Y_1 \leq X, \ldots, Y_{k-1} \leq X, Y_k \leq X)}{P(Y_1 \leq X, \ldots, Y_{k-1} \leq X, Y_k \leq X)} \\
&\leq \frac{P(X \leq d, Y_1 \leq X, \ldots, Y_{k-1} \leq X, Y_k \leq X)}{P(Y_1 \leq X, \ldots, Y_{k-1} \leq X, Y_k \leq X, Y_k \leq d)} \\
&\leq \frac{P(X \leq d, Y_1 \leq X, \ldots, Y_{k-1} \leq X, Y_k \leq X) + P(Y_k \leq d, Y_1 \leq X, \ldots, Y_{k-1} \leq X, Y_k > X)}{P(Y_1 \leq X, \ldots, Y_{k-1} \leq X, Y_k \leq X, Y_k \leq d) + P(Y_k \leq d, Y_1 \leq X, \ldots, Y_{k-1} \leq X, Y_k > X)} \\
&= \frac{P(X \leq d, Y_1 \leq X, \ldots, Y_{k-1} \leq X, Y_k \leq d)}{P(Y_1 \leq X, \ldots, Y_{k-1} \leq X, Y_k \leq d)} \\
&= \frac{P(X \leq d, Y_1 \leq X, \ldots, Y_{k-1} \leq X)P(Y_k \leq d)}{P(Y_1 \leq X, \ldots, Y_{k-1} \leq X)P(Y_k \leq d)} \\
&= \frac{P(X \leq d, Y_1 \leq X, \ldots, Y_{k-1} \leq X)}{P(Y_1 \leq X, \ldots, Y_{k-1} \leq X)} \\
&= P(X \leq d \,|\, Y_1 \leq X, \ldots, Y_{k-1} \leq X).
\end{aligned} \qquad \textbf{Equation 40.}$$

In the fourth line **Equation 39** was applied and the sixth line follows from independence. The proof for the case where $k = 1$ is an given by analogous (but simpler) calculation to the one shown above, completing the proof.[c] □

With **Lemma 7**, **Equation 36** is justified. Thus, we have

$$\begin{aligned}
&E\left[\Pi^{ST}(v, b_n(v)) \,\middle|\, \max_{1\leq i\leq n-1} b_i(V_i) \leq B_0\right] \\
&= (v - b_n(v))P(B_0 \leq b_n(v) \,|\, b_1(V_1) \leq B_0, \ldots, b_{n-2}(V_{n-2}) \leq B_0, b_{n-1}(V_{n-1}) \leq B_0) \\
&\leq (v - b_n(v))P(B_0 \leq b_n(v) \,|\, b_1(V_1) \leq B_0, \ldots, b_{n-2}(V_{n-2}) \leq B_0) \\
&= E\left[\Pi^{ST}(v, b_n(v)) \,\middle|\, \max_{1\leq i\leq n-2} b_i(V_i) \leq B_0\right].
\end{aligned} \qquad \textbf{Equation 41.}$$

Similarly, one can use **Lemma 7** to show

$$E\left[\Pi^{ST}(v, b_n(v)) \,\middle|\, \max_{1\leq i\leq 1} b_i(V_i) \leq B_0\right] \leq E\left[\Pi^{ST}(v, b_n(v))\right]. \qquad \textbf{Equation 42.}$$

We can now state a theorem on our expected surplus.

Theorem 8. *Let* $n > 1$. *Suppose the conditions of the ST game and of* ***Theorem 5*** *are satisfied and the functions* b_{n-1} *and* b_n *are defined as in* ***Theorem 5***. *Also, define* $E[\Pi^{ST}(v, b)| \max_{1\leq i\leq 0} b_i(V_i) \leq B_0]$ *as* $E[\Pi^{ST}(v, b)]$. *If* $v = 0$ *or* $v \geq \phi_{n-1}(\omega)$ *we have*

$$E\left[\Pi^{ST}(v, b_{n-1}(v)) \,\middle|\, \max_{1\leq i\leq n-2} b_i(V_i) \leq B_0\right] = E\left[\Pi^{ST}(v, b_n(v)) \,\middle|\, \max_{1\leq i\leq n-1} b_i(V_i) \leq B_0\right]. \qquad \textbf{Equation 43.}$$

If $0 < v < \phi_{n-1}(\omega)$, *then*

$$E\left[\Pi^{ST}(v, b_{n-1}(v)) \,\middle|\, \max_{1\leq i\leq n-2} b_i(V_i) \leq B_0\right] > E\left[\Pi^{ST}(v, b_n(v)) \,\middle|\, \max_{1\leq i\leq n-1} b_i(V_i) \leq B_0\right]. \qquad \textbf{Equation 44.}$$

Proof. If $v = 0$, then

$$E[\Pi^{ST}(0, b_{n-1}(0))| \max_{1\leq i\leq n-2} b_i(V_i) \leq B_0] = 0 = E[\Pi^{ST}(0, b_n(0))| \max_{1\leq i\leq n-1} b_i(V_i) \leq B_0]. \qquad \textbf{Equation 45.}$$

[c]The authors would like to thank Byungchul Cha and William Dunham for their contributions to the above proof. In particular, the central idea of the proof of **Lemma 7** is due to Dr. Cha.

If $v \geq \phi_{n-1}(\omega)$, then, as we argued in the proof of **Theorem 6**, then $v \geq \phi_{n-1}(\omega) > \phi_n(\omega)$. Thus, we have

$$E[\Pi^{ST}(v, b_{n-1}(v))| \max_{1\leq i\leq n-2} b_i(V_i) \leq B_0] = v - \omega = E[\Pi^{ST}(v, b_n(v))| \max_{1\leq i\leq n-1} b_i(V_i) \leq B_0]. \quad \textbf{Equation 46.}$$

Finally, suppose $0 < v < \phi_{n-1}(\omega)$. Using **Equation 41** if $n > 2$ or **Equation 42** if $n = 2$, we have

$$\begin{aligned} E\left[\Pi^{ST}(v, b_n(v)) \middle| \max_{1\leq i\leq n-1} b_i(V_i) \leq B_0\right] &\leq E\left[\Pi^{ST}(v, b_n(v)) \middle| \max_{1\leq i\leq n-2} b_i(V_i) \leq B_0\right] \\ &< E\left[\Pi^{ST}(v, b_{n-1}(v)) \middle| \max_{1\leq i\leq n-2} b_i(V_i) \leq B_0\right], \quad \textbf{Equation 47.} \end{aligned}$$

where in the last line we used the fact that $b_{n-1}(v)$ is the unique maximizer of

$$E\left[\Pi^{ST}(v, b) \middle| \max_{1\leq i\leq n-2} b_i(V_i) \leq B_0\right] \quad \textbf{Equation 48.}$$

by either **Theorem 5** if $n > 2$ or by **Theorem 4** if $n = 2$, and $b_{n-1}(v) \neq b_n(v)$ by **Theorem 6**. This completes the proof. □

Heuristically, **Theorem 8** says that a bidder can expect more surplus from a ST game in which fewer failed bids have been made. This is despite the fact that a bidder who bids after n failed bids has more information than the $n-1^{st}$ bidder. The extra information that $b_{n-1}(V_{n-1}) \leq B_0$ is bad news for all bidders, as it signals that B_0 is potentially high.

SERIAL FIRST-PRICE BUY-OUT AUCTION

In this section we consider a model which we will call the *Serial First-Price Buy-Out (SFPBO) auction*. We will give our formal mathematical assumptions below, but we first describe this auction informally. In this auction, bidders place bids in a first-price setting (that is, the highest bidder wins the item and pays her bid). Like the ST game, there is a "buy-out" price ω (which plays the role of a BIN price) which is the maximum allowed bid for the auction. Unlike the ST game, in the SFPBO auction bidders are directly competing against one another as the seller will accept the highest bid submitted among all bidders. Bidders bid one at a time in an unknown order, and any bidder who submits the buy-out price ω for a bid will win the auction with certainty and end the auction. Bidders do not know how many previous bids have been made and do not know how many bids will be. Instead, they know a distribution of possible competitors and we will denote the probability of n competing bidders as q_n using the notation of Krishna.[5] Although the bidder who has the chance to bid does not know how many bidders have bid before her, she does know that no previous bidder has submitted the buy-out price (otherwise, the auction would have ended) and this fact will be built into our definition of Nash equilibria for the SFPBO model. We will model the SFPBO auction in one round where each bidder places a bid between 0 and the buy-out price of ω. Below we list our mathematical assumptions for the SFPBO auction.

Definition 9. *The conditions of the SFPBO auction are*

(i) Fix a number $m \in \mathbb{N}$. We suppose $m+1$ is the maximum amount of possible total bidders in the auction. For each $i \in \{1, \ldots, m+1\}$, the i^{th} possible bidder's value for the item for auction is represented by the random variable V_i with range $[0,1]$. Each V_i has a cdf $F(v)$ and a pdf $f(v)$. Note that this means all potential bidder values are identically distributed. Let $\vec{V} = (V_1, V_2, \ldots, V_{m+1})$.

(ii) The pdf satisfies $f(v) > 0$ for all $v \in (0,1)$. In particular, the cdf is strictly increasing.

(iii) For all $i \in \{1, 2, \ldots, m+1\}$, there is a Bernoulli random variable X_i that equals 0 if the i^{th} possible bidder is not an active bidder and 1 if the i^{th} possible bidder is an active bidder. We define B to be the index set of active bidders. That is,

$$B = \{j \in \{1, \ldots, m+1\} : X_j = 1\}. \quad \textbf{Equation 49.}$$

Let $\vec{X} = (X_1, X_2, \ldots, X_{m+1})$. We assume that $X_1, \ldots, X_{m+1}$ satisfy the following symmetry condition: For any $\vec{\beta} \in \{0,1\}^{m+1}$ and permutation σ on $m+1$ elements, we have

$$P\big[(X_1, X_2, \ldots, X_{m+1}) = (\beta_1, \beta_2, \ldots, \beta_{m+1})\big] = P\big[(X_1, X_2, \ldots, X_{m+1}) = (\beta_{\sigma(1)}, \beta_{\sigma(2)}, \ldots, \beta_{\sigma(m+1)})\big].$$

Equation 50.

In particular, note by symmetry, $X_1, X_2, \ldots, X_{m+1}$ are identically distributed. Also note that we are allowing for the possibility that $X_1, X_2, \ldots, X_{m+1}$ are correlated.

(iv) *Define $p := P(X_1 = 1)$. Then $p = P(X_i = 1)$ for all $i \in \{1, \ldots, m+1\}$. We also impose the condition $p > 0$ so that there is a positive probability that bidders will bid in the auction.*

(v) *Let Σ be a random permutation on $\{1, \ldots, m+1\}$ with a "uniform distribution." That is, for any permutation σ on $\{1, \ldots, m+1\}$, $P(\Sigma = \sigma) = 1/(m+1)!$. We think of Σ as giving the bidders a random order in which to bid in the auction, or equivalent, establishing an order for tie-breaking.*

(vi) *For any $j \in \{1, \ldots, m+1\}$, the random variables $V_1, V_2, \ldots V_{m+1}, \Sigma, X_j$ are independent.*

(vii) *Let $N := X_1 + X_2 + \ldots + X_{m+1} = |B|$. That is, N is the number of bidders participating in the auction. For all $n \in \{1, 2, \ldots m+1\}$, let $\tilde{q}_n = P(N = n)$. That is, $\tilde{q}_n$ is the probability distribution of N. For all $n \in \{0, 1, \ldots m\}$, define q_n as $q_n = P(N = n+1 | X_1 = 1)$. By the symmetry condition stated in assumption (iii), note that for all $i \in \{1, \ldots, m+1\}$ we have $q_n = P(N = n+1 | X_i = 1)$. That is, for each active bidder q_n is the probability that bidder is facing n opponents.*

(viii) *For each $i \in \{1, \ldots, m+1\}$, define the random variable $Y_i = \max_{1 \le j \le m+1,\, i \ne j} X_j V_j$. Let $G_i(y)$ denote the cdf of each Y_i dependent on the event $X_i = 1$. That is,*

$$G_i(y) := P(Y_i \le y | X_i = 1).$$

Equation 51.

Also define the random variable $\tilde{Y} = \max_{1 \le j \le m+1} X_j V_j$ and let $\tilde{G}(y)$ denote its cdf.

(ix) *There exists a maximum bid ω such that $0 < \omega \le 1$ above which no bids can be placed. That is, all bids must lie in the range $[0, \omega]$.*

(x) *For any $i \in \{1, \ldots, n\}$ and vector $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, define $\vec{y}_{-i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ (that is, $\vec{y}_{-i}$ is the vector $\vec{y}$ with the i^{th} entry omitted). For any function $f : \mathbb{R} \to \mathbb{R}$ and vector $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, define $f(\vec{y}) = (f(y_1), f(y_2), \ldots, f(y_n))$.*

(xi) *Let $i \in \{1 \ldots, m+1\}$. If i^{th} possible bidder is actually a bidder and gets the opportunity to bid, then she submits a bid b_i. The surplus for the i^{th} possible bidder is*

$$\Pi_i(V_i, b_i, \vec{b}_{-i}, \vec{X}_{-i}, \Sigma) = (V_i - b_i) \mathbb{1}_{b_i > \max\{b_j : j \in B, \Sigma(j) < \Sigma(i)\}} \mathbb{1}_{b_i \ge \max\{b_j : j \in B, \Sigma(j) > \Sigma(i)\}},$$

Equation 52.

where we define the maximum over the empty set to be $-\infty$.

(xii) *Bidder i will only have the opportunity to bid if $\Sigma(i) = 1$ or $\Sigma(i) > 1$ and none of bidders $\Sigma^{-1}(1)$ through $\Sigma^{-1}(\Sigma(i) - 1)$ bid the maximum bid of ω. Any bidder who bids the maximum bid of ω automatically wins the auction.*

First we elaborate on some of the assumption of the SFPBO auction. In the definition of Π_i in assumption (xi), we see that the permutation Σ determines the tie-breaking order. That is, bidder $\Sigma^{-1}(1)$ wins all ties she is involved with, while $\Sigma^{-1}(2)$ wins all ties except against bidder $\Sigma^{-1}(1)$, etc. We will see that these ties will occur with probability 0 except for when multiple bidders submit the maximum price of ω. That is, tie-breaking will only be important when multiple bidders bid ω. This assumption, along with assumptions (v) and (xii) highlight the fact that bids are submitted one at a time in a serial fashion according to Σ and that any bidder who agrees to pay the maximum price of ω will win the auction and thereby end the auction. In particular, assumption (xii) means that a bidder actually has some extra information about previous bidding behavior just from the fact that she has the opportunity to bid. Not only does such a bidder know that she will win the auction with certainty if she submits the maximum bid ω, but she can also infer some information about the previous bidders values if all competing bidders are using a particular bidding function. We will build this fact into our definition of equilibrium strategies below in **Definitions 10** and **11**.

Before continuing onto the mathematics of the SFPBO auction, we should state here how our model relates to the eBay Best Offer auction. In the eBay best offer auction, potential bidders come across the auction at different times and can decide to end the auction by agreeing to pay the Buy-It-Now price. Our SFPBO auction accounts for the serial nature of this process, as any bidder who has a chance to submit a bid can win the auction instantly by submitting the maximum price ω as a bid (see assumption (xii)). The serial nature of this model does require an additional layer of complication in the form of the random permutation Σ as defined in assumption (v). Also on eBay, bidders do not know how many bidders have submitted bids before them nor how many bidders will submit bids after them. We build this into the SFPBO auction with the unknown number of bidders (see assumptions (iii) and (vii)).

The SFPBO auction differs from the eBay Best Offer auction in two key ways. The first difference is that our model assumes that if no bidder bids the Buy-It-Now price, then the seller will accept the highest offer given. This need not be the case, as the seller has the ability to reject all offers and keep the item. Furthermore, offers on eBay have a time limit of 48 hours. Thus, the best offer the seller receives may have expired by the time the seller decides to accept that offer. The SFPBO auction does not model any of this decision-making required by an actual seller (except for the setting of the maximum bid of ω). The second key difference is that in the SFPBO auction (as in the ST game), bidders can submit up to three offers for the same item, and a seller can counteroffer the two first offers made by a bidder. Our model does not account for this bargaining, although the third and final (and uncounterable) offer could be seen as the single bid submitted in the SFPBO auction.

Next we want to classify what sort of bidding functions $b(v)$ we are allowing for the SFPBO auction. As in the work of Reynolds and Wooders, we consider a *cut-off* strategy. That is, we should have a strategy that is increasing in bidder values up to a "cut-off" point v_* after which bidders will bid the buy-out price of ω. Specifically, we have the following definition (which also places some smoothness conditions on our bidding function).

Definition 10. *Assume the conditions of the SFPBO auction. Consider a function $b : [0,1] \to [0,\omega]$ and define $v_*(b)$ (or simply v_*) as*

$$v_*(b) = v_* := \inf\left(\{v : b(v) = \omega\} \cup \{1\}\right). \qquad \textbf{Equation 53.}$$

We say that b is a permissible bidding function for the SFPBO auction *(or simply* permissible*) if*

(a) b is nondecreasing on $[0,1]$, and increasing on $[0,v_*]$, and

(b) b is continuous on $[0,v_*]$ and differentiable on $(0,v_*)$.

Note that any permissible bidding function with $v_* < 1$ is constantly equal to ω on $(v_*,1]$. As with most auctions, we cannot hope to find a so-called *dominant* strategy that maximizes the expected surplus no matter how our opponents bid. The best we can hope for is a type of Nash equilibrium strategy where if all the bidder's opponents use the equilibrium strategy, then the strategy that maximizes the expected profit of the bidder is the same equilibrium strategy. In light of assumption (xii), our equilibrium will actually maximize a bidder's expected value conditioned on the event that bidder had the opportunity to bid.

Definition 11. *Assume the conditions of the SFPBO auction. For any $v \in [0,1]$ and $i \in \{1,\dots,m+1\}$, define the event C_v^i as*

$$C_v^i = \{X_j = 0 \text{ or } V_j \le v \text{ for all } j \in \{1,\dots,m+1\} \text{ such that } \Sigma(j) < \Sigma(i)\}. \qquad \textbf{Equation 54.}$$

That is, C_v^i is the event that all active bidders who bid before bidder i in the SFPBO auction had a value less than v.

We say that $b : [0,1] \to [0,\omega]$ *is a* permissible symmetric Bayesian Nash equilibrium bidding strategy for the SFPBO auction *(or just* permissible SBNE*) if* b *is permissible and for all* $i \in \{1,\ldots,m+1\}$

$$\sup_{\hat{b}\in[0,\omega]} E\left[\Pi_i(v_i,\hat{b},b(\vec{V}_{-i}),\vec{X}_{-i},\Sigma)\middle| C^i_{v_*(b)}, X_i = 1\right] = E\left[\Pi_i(v_i,b(v_i),b(\vec{V}_{-i}),\vec{X}_{-i},\Sigma)\middle| C^i_{v_*(b)}, X_i = 1\right]. \quad \textbf{Equation 55.}$$

Note that our definition of a permissible SBNE includes a conditional expectation to reflect that bidder i knows that all bidders who bid before her who used the bidding function b must have had a value below the cut-off value $v_*(b)$ (hence conditioning on the event $C^i_{v_*}$) as well as the fact that she is an active bidder (hence conditioning on the event $X_i = 1$). Given the presence in the definition of a permissible SBNE, we next find the probability of the event C^i_v given $X_i = 1$.

Proposition 12. *Assume the conditions of the SFPBO auction. Let* $v \in [0,1]$ *and* $i \in \{1,\ldots,m+1\}$. *Then*

$$P(C^i_v|X_i = 1) = \sum_{n=0}^{m} \frac{1}{n+1}\left(\sum_{k=0}^{n} F(v)^k\right) q_n = \begin{cases} \sum_{n=0}^{m} \frac{1}{n+1}\frac{1-F(v)^{n+1}}{1-F(v)} q_n & \textit{if } v < 1 \\ 1 & \textit{if } v = 1. \end{cases} \quad \textbf{Equation 56.}$$

In particular, $P(C^i_v|X_i = 1)$ *does not depend on* i *and is differentiable in* v.

Proof. If we know that $N = n+1$ for some fixed $n \in \{0,\ldots,m\}$, then we know that random set of bidders B has $n+1$ elements. If $X_i = 1$, applying the random permutation Σ to the elements of B can put the bidder i in any position relative to the other active bidders in B with equal probability (specifically, with probability $\frac{1}{(n+1)}$). So, since Σ is independent of $\vec{V}$ and each X_j we have

$$\begin{aligned} P(C^i_v|X_i = 1, N = n+1) &= P(V_j \le v \text{ for all } j \in B \text{ such that } \Sigma(j) < \Sigma(i)|X_i = 1, N = n+1) \\ &= \sum_{k=0}^{n} F(v)^k \frac{1}{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} F(v)^k \\ &= \begin{cases} \frac{1}{n+1}\frac{1-F(v)^{n+1}}{1-F(v)} & \text{if } v < 1 \\ 1 & \text{if } v = 1. \end{cases} \end{aligned} \quad \textbf{Equation 57.}$$

Thus, by the rule of total probability we have

$$\begin{aligned} P(C^i_v|X_i = 1) &= \sum_{n=0}^{m} P(C^i_v|X_i = 1, N = n+1)P(N = n+1|X_i = 1) \\ &= \sum_{n=0}^{m} P(C^i_v|X_i = 1, N = n+1) q_n \\ &= \sum_{n=0}^{m} \frac{1}{n+1}\left(\sum_{k=0}^{n} F(v)^k\right) q_n \\ &= \begin{cases} \sum_{n=0}^{m} \frac{1}{n+1}\frac{1-F(v)^{n+1}}{1-F(v)} q_n & \text{if } v < 1 \\ 1 & \text{if } v = 1, \end{cases} \end{aligned} \quad \textbf{Equation 58.}$$

proving the result. □

Next we will derive a unique permissible SBNE for the SFPBO auction by standard auction theory techniques. We will work backwards by first assuming such a permissible SBNE exists and deriving a specific formula for that permissible SBNE. Then will we prove that this formula actually yields a permissible SBNE. First we start with the value of b at 0.

Proposition 13. *Assume the conditions of the SFPBO auction. Let* $b : [0,1] \to [0,\omega]$ *be an permissible SBNE for the SFPBO auction. Then* $b(0) = 0$.

Proof. First note that a bidder can only bid nonnegative values. That is, $b(0) \geq 0$. In order to show $b(0) = 0$, we will proceed by contradiction. Let's assume $b(0) \neq 0$. Thus, we have $b(0) > 0$. Since $\Pi_1\big(0, b(0), b(\vec{V}_{-1}, \vec{X}_{-1}, \Sigma))\big)$ will always be negative or zero, we can write

$$\Pi_1\big(0, 0, b(\vec{V}_{-1}, \vec{X}_{-1}, \Sigma))\big) = 0 \geq \Pi_1\big(0, b(0), b(\vec{V}_{-1}, \vec{X}_{-1}, \Sigma))\big). \quad \text{Equation 59.}$$

Also,

$$\begin{aligned} P\left(\Pi_1\left(0, b(0), b(\vec{V}_{-1}), \vec{X}_{-1}, \Sigma\right) < 0\right) &= P\left(b(0) \geq \max_{2\leq i\leq m+1} X_i b(V_i)\right) \\ &= P\left(X_i = 1 \text{ and } V_i > 0 \text{ for some } i \in \{2, ..., m+1\}\right) \\ &> 0. \end{aligned} \quad \text{Equation 60.}$$

Thus $P\left(\Pi_1\left(0, b(0), b(\vec{V}_{-1}), \vec{X}_{-1}, \Sigma\right) < 0\right)$ is positive. To put it into words, we have a positive probability of getting a negative surplus when bidding $b(0)$.

So $\Pi_1\big(0, b(0), b(\vec{V}_{-1}), \vec{X}_{-1}, \Sigma\big) \leq 0$ and is strictly less than 0 with positive probability. It follows that,

$$E\Big[\Pi_1\big(0, b(0), b(\vec{V}_{-0}), \vec{X}_{-1}, \Sigma\big)\big|C^1_{v_*}, X_1 = 1\Big] < 0 = E\Big[\Pi_1\big(0, 0, b(\vec{V}_{-0}), \vec{X}_{-1}, \Sigma\big)\big|C^1_{v_*}, X_1 = 1\Big]. \quad \text{Equation 61.}$$

Therefore, bidding zero will result in a higher expected surplus than bidding $b(0)$. This contradicts our assumption that b is an permissible SBNE. Thus, $b(0) \not> 0$. Thus, as a bidder can only bid nonnegative values we have $b(0) = 0$ as desired. □

Also, we can show that the cut-off value v_* must be positive.

Proposition 14. *Assume the conditions of the SFPBO auction. Let $b : [0, 1] \to [0, \omega]$ be a permissible SBNE for the SFPBO auction with the associated value v_*. Then $v_* > 0$.*

Proof. Note that $E\left[\Pi_1(\omega/2, \omega, b(\vec{V}_{-1}), \vec{X}_{-1}, \Sigma)\middle| C^1_{v_*}, X_1 = 1\right] = -\omega/2 < 0 = E\left[\Pi(\omega/2, 0, b(\vec{V}_{-1}, \vec{X}_{-1}, \Sigma)\middle| C^1_{v_*}, X_1 = 1\right]$. Thus, since b is a permissible SBNE, $b(\omega/2) \neq \omega$. Since b is nondecreasing, this proves that

$$0 < \omega/2 \leq \inf\left(\{v : b(v) = \omega\} \cup \{1\}\right) = v_*, \quad \text{Equation 62.}$$

as desired. □

The next part of our plan is find a formula for a permissible SBNE $b(v)$ over the domain $[0, v_*]$. To do this, we will need to compute the conditional cdf $G_i(y)$ of the first order statistic random variable Y_i. To that end, we have the following definition and proposition.

Definition 15. *Assume the conditions of the SFPBO auction. Define the function $G : [0, 1] \to [0, 1]$ as*

$$G(y) = \sum_{n=0}^{m} F(y)^n q_n. \quad \text{Equation 63.}$$

Proposition 16. *Assume the conditions of the SFPBO auction. For each $i \in \{1, \ldots, m+1\}$ the conditional cdf $G_i(y) = P(Y_i \leq y|X_i = 1)$ is given by*

$$G_i(y) = G(y) = \sum_{n=0}^{m} F(y)^n q_n. \quad \text{Equation 64.}$$

In particular, G_i does not depend on i.

Proof. Let $i \in \{1, \ldots, m+1\}$ be arbitrary. Then we have

$$\begin{aligned} G_i(y) &= P(Y_i \leq y | X_i = 1) \\ &= \sum_{n=0}^{m} P(Y_i \leq y | N = n+1, X_i = 1) P(N = n+1 | X_i = 1) \\ &= \sum_{n=0}^{m} F(y)^n q_n, \end{aligned} \qquad \textbf{Equation 65.}$$

as desired. □

Now we can compute a formula for the bidder's expected surplus.

Proposition 17. *Assume the conditions of the SFPBO auction. Let $b : [0,1] \to [0,\omega]$ be a permissible bidding function for the SFPBO auction with the cut-off value of v_*, and suppose that $b(0) = 0$. Let b^{-1} be the inverse of $b|_{[0,v_*]}$ and fix $i \in \{1, \ldots, m+1\}$. If $v \in [0,1]$ and $\hat{b} \in [0,\omega]$, then*

$$E\Big[\Pi_i\big(v, \hat{b}, b(\vec{V}_{-i}), \vec{X}_{-i}, \Sigma\big)\Big| C^i_{v_*}, X_i = 1\Big] = \begin{cases} \frac{(v-\hat{b})G(b^{-1}(\hat{b}))}{P(C^i_{v_*}|X_1=1)} & \text{if } \hat{b} \leq b(v_*) \text{ and } \hat{b} < \omega \\ \frac{(v-\hat{b})G(v_*)}{P(C^i_{v_*}|X_1=1)} & \text{if } b(v_*) < \hat{b} < \omega \\ v - \omega & \text{if } \hat{b} = \omega. \end{cases} \qquad \textbf{Equation 66.}$$

In particular, the above equation holds if b is a permissible SBNE and $E\left[\Pi_i\big(v, \hat{b}, b(\vec{V}_{-i}), \vec{X}_{-i}, \Sigma\big)\middle| C^i_{v_*}, X_i = 1\right]$ is independent of i.

We could rewrite **Equation 66** using **Proposition 12** but refrain from doing so at the moment for the sake of compactness.

Proof. Note that if $\hat{b} = \omega$, bidder i will win the auction with certainty with surplus $v - \omega$ proving **Equation 66** in the case $\hat{b} = \omega$. To prove the other two cases, suppose $\hat{b} < \omega$. Thus, for any j we have $P(b(V_j) = \hat{b}) = 0$, so we have

$$\begin{aligned} & E\left[\Pi_i\big(v, \hat{b}, b(\vec{V}_{-i}), \vec{X}_{-i}, \Sigma\big) \,|C^i_{v_*}, X_i = 1\right] \\ = \quad & (v - \hat{b}) E\left[\mathbb{1}_{\hat{b} > \max\{b(V_j): j \in B, \Sigma(j) < \Sigma(i)\}} \mathbb{1}_{\hat{b} \geq \max\{b(V_j): j \in B, \Sigma(j) > \Sigma(i)\}} \,|C^i_{v_*}, X_i = 1\right] \\ = \quad & (v - \hat{b}) P\left(\hat{b} \geq \max\{b(V_j) : j \in B\} \,|C^i_{v_*}, X_i = 1\right) \\ = \quad & (v - \hat{b}) P\left(\hat{b} \geq \max\{X_j b(V_j) : 1 \leq j \leq m+1, j \neq i\} \,|C^i_{v_*}, X_i = 1\right) \\ = \quad & (v - \hat{b}) P\left(\hat{b} \geq \max\{b(X_j V_j) : 1 \leq j \leq m+1, j \neq i\} \,|C^i_{v_*}, X_i = 1\right), \end{aligned} \qquad \textbf{Equation 67.}$$

where on the last line we used the fact that $b(0) = 0$. Continuing **Equation 67**, we have

$$\begin{aligned} & E\left[\Pi_i\big(v, \hat{b}, b(\vec{V}_{-i}), \vec{X}_{-i}, \Sigma\big) \,|C^i_{v_*}, X_i = 1\right] \\ = \quad & (v - \hat{b}) P\left(\hat{b} \geq \max\{b(X_j V_j) : 1 \leq j \leq m+1, j \neq i\} \,|C^i_{v_*}, X_i = 1\right) \\ = \quad & (v - \hat{b}) P\left(\hat{b} \geq b(Y_i) \,|C^i_{v_*}, X_i = 1\right) \\ = \quad & (v - \hat{b}) \frac{P\left(\hat{b} \geq b(Y_i), C^i_{v_*}, X_i = 1\right)}{P(C^i_{v_*}, X_i = 1)}. \end{aligned} \qquad \textbf{Equation 68.}$$

Since $\hat{b} < \omega$ by assumption, the event $\hat{b} \geq b(Y_i)$ implies the event $Y_i \leq v_*$ which in turn implies the event $C^i_{v_*}$. By this observation, as

$$\begin{aligned} E\left[\Pi_i\big(v, \hat{b}, b(\vec{V}_{-i}), \vec{X}_{-i}, \Sigma\big)\,\middle|C^i_{v_*}, X_i = 1\right] &= (v-\hat{b})\frac{P\left(\hat{b} \geq b(Y_i), C^i_{v_*}, X_i = 1\right)}{P(C^i_{v_*}, X_i = 1)} \\ &= (v-\hat{b})\frac{P\left(\hat{b} \geq b(Y_i), X_i = 1\right)}{P(C^i_{v_*}, X_i = 1)} \\ &= (v-\hat{b})\frac{P\left(\hat{b} \geq b(Y_i)|X_i = 1\right)}{P(C^i_{v_*}|X_i = 1)} \\ &= \begin{cases} (v-\hat{b})\frac{P\left(b^{-1}(\hat{b}) \geq Y_i|X_i=1\right)}{P(C^i_{v_*}|X_i=1)} & \text{if } \hat{b} \leq b(v_*) \text{ and } \hat{b} < \omega \\ (v-\hat{b})\frac{P(v_* \geq Y_i|X_i=1)}{P(C^i_{v_*}|X_i=1)} & \text{if } b(v_*) < \hat{b} < \omega \end{cases} \\ &= \begin{cases} \frac{(v-\hat{b})G(b^{-1}(\hat{b}))}{P(C^i_{v_*}|X_1=1)} & \text{if } \hat{b} \leq b(v_*) \text{ and } \hat{b} < \omega. \\ \frac{(v-\hat{b})G(v_*)}{P(C^1_{v_*}|X_1=1)} & \text{if } b(v_*) < \hat{b} < \omega \end{cases}, \end{aligned} \quad \textbf{Equation 69.}$$

where we used **Proposition 12** and **Proposition 16** in the last line. This completes the proof. □

The above proposition will allow us to derive a formula for the equilibrium bidding function $b(v)$ over $[0, v_*]$. However, before we derive such a formula, we note $(v-\hat{b})\frac{P\left(\hat{b} \geq b(Y_i), C^i_{v_*}, X_i=1\right)}{P(C^i_{v_*}, X_i=1)} = (v-\hat{b})\frac{P\left(\hat{b} \geq b(Y_i), X_i=1\right)}{P(C^i_{v_*}, X_i=1)}$ above. This equation tells us that for any bid $\hat{b} < \omega$, the fact the event $C^i_{v_*}$ has occurred does not affect the probability of the event of having the highest bid among all opponents (i.e. the event $\hat{b} \geq b(Y_i)$). That is, a bidder who bid an amount $\hat{b}$ that is less than the buy-out price ω does not need to take into account that any previous bidders must have a value below the cut-off of v_*, because for her to win the auction *all* bidders must have values below the cut-off value. This fact will mean that the conditional nature of the expected value on $C^i_{v_*}$ will not profoundly affect the equilibrium behavior for bidders whose values fall below the cut-off v_*. Thus, one could expect that the equilibrium found in an auction with simultaneous bidding and a random tie-breaker for bidders who bid ω is the same as one found in a SFPBO auction. We will show that this is indeed the case in the next section. But now we return to deriving the formula for the equilibrium bidding function over $[0, v_*]$ for the SFPBO auction.

Theorem 18. *Assume the conditions of the SFPBO auction. Let* $b : [0, 1] \to [0, \omega]$ *be an permissible SBNE for the SFPBO auction. Then, if* $v \in [0, v_*]$

$$b(v) = \begin{cases} v - \dfrac{\int_0^v G(y)dy}{G(v)} & \textit{if } 0 < v \leq v_* \\ 0 & \textit{if } v = 0. \end{cases} \quad \textbf{Equation 70.}$$

The derivation of **Equation 70** is entirely analogous to the derivation of the SBNE of a first-price sealed-bid auction and that derivation is well-known.[2, 5, 6] We still give a proof of **Equation 70** for the sake of self-containment.

Proof. Let $x, y \in (0, v_*)$. By **Proposition 17**,

$$E\left[\Pi_i\big(y, b(x), b(\vec{V}_{-1}), \vec{X}_{-1}, \Sigma\big)\middle|C^1_{v_*}, X_1 = 1\right] = \frac{\big(y - b(x)\big)G(x)}{P(C^1_{v_*}|X_1 = 1)}. \quad \textbf{Equation 71.}$$

Then, taking the derivative of our expected surplus with respect to x, we have:

$$\tfrac{d}{dx}E\left[\Pi\big(y, b(x), b(\vec{V}_{-0})\big)\right] = \frac{1}{P(C^1_{v_*}|X_1 = 1)}\Big(\big(y - b(x)\big)G'(x) - b'(x)G(x)\Big). \quad \textbf{Equation 72.}$$

Since b is an permissible SBNE, we know $E\Big[\Pi_i\big(y, b(x), b(\vec{V}_{-1}), \vec{X}_{-1}, \Sigma\big)\Big|C^1_{v_*}, X_1 = 1\Big]$ is maximized when $x = y$. Thus we must have a critical point at $x = y$. So,

$$0 = yG'(y) - b(y)G'(y) - b'(y)G(y). \quad \text{Equation 73.}$$

Adding $b(y)G'(y) + b'(y)G(y)$ to both sides to the above equation yields

$$yG'(y) = G(y)b'(y) + G'(y)b(y) = \frac{d}{dy}\Big[G(y)b(y)\Big] \text{ for all } y \in (0, v_*). \quad \text{Equation 74.}$$

Then, for any $v \in (0, v_*]$, we can integrate both sides of the above from 0 to v and have:

$$G(v)b(v) - G(0)b(0) = \int_0^v \frac{d}{dy}\left(G(y)b(y)\right)dy = \int_0^v yG'(y)dy. \quad \text{Equation 75.}$$

Note that $b(0) = 0$ by **Proposition 13**, and so we can rewrite **Equation 75** as:

$$G(v)b(v) = \int_0^v yG'(y)dy. \quad \text{Equation 76.}$$

Integrating by parts, we have

$$G(v)b(v) = yG(y)\Big|_0^v - \int_0^v G(y)dy = vG(v) - \int_0^v G(y)dy. \quad \text{Equation 77.}$$

Let $v > 0$. Then $G(v) > 0$ and we can divide both sides of our function by $G(v)$ to get $b(v) = v - \frac{\int_0^v G(y)dy}{G(v)}$. Since $b(0) = 0$ by **Proposition 13**, this completes the proof. □

Theorem 18 gives us a necessary formula for our permissible SBNE over $[0, v_*]$. This formula would look familiar to anyone who has studied auction theory, as it is the standard SBNE formula for a first-price sealed-bid auction with our particular conditional cdf $G(y)$ for the first order statistic Y_i. Our SFPBO auction is different as the maximum bid is capped by ω, and thus we have to reckon with how we can find v_*, the value above which bidders will just use the buy-out price ω. To that end, we define a function inspired by **Theorem 18**. Define $\tilde{b} : [0, 1] \to [0, 1]$ as

$$\tilde{b}(v) := \begin{cases} v - \dfrac{\int_0^v G(y)dy}{G(y)} & \text{if } 0 < v \leq 1 \\ 0 & \text{if } v = 0. \end{cases} \quad \text{Equation 78.}$$

$\tilde{b}$ would be the SBNE for a first-price sealed-bid auction with no reserve, an unknown number of bidders, and no buy-out price.[5] The content of **Theorem 18** can be restated as: if b is a permissible SBNE for the SFPBO auction, then for all $v \in [0, v_*]$ we have $b(v) = \tilde{b}(v)$. To characterize v_*, we define a new function Δ that is meant to measure the difference in expected surplus to a bidder by bidding $\tilde{b}(v)$ or bidding ω.

Definition 19. *Assume the conditions of the SFPBO auction. Define the function* $\Delta : [0, 1] \to \mathbb{R}$ *by*

$$\Delta(v) := (v - \tilde{b}(v))\frac{G(v)}{P(C^1_v|X_1 = 1)} - (v - \omega) = \frac{\int_0^v G(y)dy}{P(C^1_v|X_1 = 1)} - v + \omega \quad \text{Equation 79.}$$

Note that by **Proposition 12**, we have

$$\Delta(v) = \begin{cases} \dfrac{(1 - F(v))\int_0^v G(y)dy}{\sum_{n=0}^m \frac{1}{n+1}(1 - F(v)^{n+1})q_n} - v + \omega & \text{if } v < 1 \\ \int_0^1 G(y)dy - 1 + \omega & \text{if } v = 1. \end{cases} \quad \text{Equation 80.}$$

Again, Δ is meant to measure which of the bidding options (making an offer or using the buy-out price) result in the most expected surplus for the bidder. If $\Delta(v) > 0$, then the bidder should choose the first option and if $\Delta(v) < 0$, then she should choose the second. If $\Delta(v) = 0$, she should be indifferent to either option. Thus, we expect that $\Delta(v_*) = 0$ if $v_* < 1$, as this is the value where the bidder switches from the first option to the second. Below, we will prove these heuristic observations.

Proposition 20. *Assume the conditions of the SFPBO auction and define* Δ *as in* ***Definition 19****. Then* Δ *is continuous on* $[0,1]$*, differentiable on* $(0,1)$*,* $\Delta(0) > 0$ *and* Δ *is decreasing on* $[0,1]$*.*

To prove the above, we will need an auxillary definition and a lemma:

Definition 21. *Let* n *be a natural number. Define the polynomial* $j_n(x)$ *as*

$$j_n(x) := -(n-1)x^n + nx^{n-1} - 1. \qquad \text{Equation 81.}$$

Lemma 22. *For any natural number* n*,* $j_n(x) = -(n-1)x^n + nx^{n-1} - 1$ *is negative for all* $x \in [0,1)$*.*

Proof. First note that $j_n(0) = -1 < 0$ and $j_n(1) = 0$. Thus, to prove the lemma it suffices to show that $j_n'(x) > 0$ for $x \in (0,1)$. To that end, note that $j_n'(x) = n(n-1)x^{n-2}(1-x) > 0$ for $x \in (0,1)$, completing the proof. □

Now we are ready to prove our proposition involving Δ:

Proof. (Proposition 20) First note that we can rewrite **Equation 80** as

$$\Delta(v) = \frac{\int_0^v G(y)dy}{\sum_{n=0}^m \frac{1}{n+1}\left(\sum_{k=0}^n F(v)^k\right) q_n} - v + \omega, \qquad \text{Equation 82.}$$

and thus Δ is continuous on $[0,1]$ and differentiable on $(0,1)$ by the Fundamental Theorem of Calculus. Next note $\Delta(0) = \omega > 0$. To prove Δ is decreasing, we simply prove its derivative is negative on $(0,1)$. Thus, for $v \in (0,1)$, we have

$$\begin{aligned}
\Delta'(v) &= \frac{d}{dv}\left(\frac{(1-F(v))\int_0^v G(y)dy}{\sum_{n=0}^m \frac{1}{n+1}(1-F(v)^{n+1})q_n} - v + \omega\right) \\
&= \frac{\sum_{n=0}^m \left(\frac{1}{n+1}(1-F(v)^{n+1})q_n\right)\left(-f(v)\int_0^v G(y)dy + (1-F(v))G(v)\right) + (1-F(v))F(v)^n f(v) q_n \int_0^v G(y)dy}{\left(\sum_{n=0}^m \frac{1}{n+1}(1-F(v)^{n+1})q_n\right)^2} - 1 \\
&= \frac{-f(v)\int_0^v G(y)dy \sum_{n=0}^m \frac{q_n}{n+1}(1-F(v)^{n+1} - (n+1)(1-F(v))F(v)^n)}{\left(\sum_{n=0}^m \frac{1}{n+1}(1-F(v)^{n+1})q_n\right)^2} + \\
&\quad \frac{\sum_{n=0}^m \frac{q_n}{n+1}((n+1)(1-F(v))F(y)^n - (1-F(v)^{n+1}))}{\sum_{n=0}^m \frac{1}{n+1}(1-F(v)^{n+1})q_n} \\
&= \frac{-f(v)\int_0^v G(y)dy \sum_{n=0}^m \frac{q_n}{n+1}(-j_n(F(v)))}{\left(\sum_{n=0}^m \frac{1}{n+1}(1-F(v)^{n+1})q_n\right)^2} - \frac{\sum_{n=0}^m \frac{q_n}{n+1}(-j_n(F(v)))}{\sum_{n=0}^m \frac{1}{n+1}(1-F(v)^{n+1})q_n} \\
&< 0
\end{aligned} \qquad \text{Equation 83.}$$

for $v \in (0,1)$ (applying **Lemma 22** in the last inequality). This completes the proof. □

Next we will show that $\Delta(v)$ is non-negative for $v \leq v_*$:

Proposition 23. *Assume the conditions of the SFPBO auction, suppose that there exists a permissible SBNE b for this auction, and define Δ as in **Definition 19**. Then $\Delta(v_*) \geq 0$ and $\Delta(v) > 0$ for all $v \in [0, v_*)$.*

Proof. Let $v \in [0, v_*]$. Using **Proposition 17** and **Theorem 18** we have

$$\begin{aligned}
\Delta(v) &= (v - \tilde{b}(v))\frac{G(v)}{P(C_v^1|X_1 = 1)} - (v - \omega) \\
&= (v - b(v))\frac{G(v)}{\sum_{n=0}^{m} \frac{1}{n+1} \left(\sum_{k=0}^{n} F(v)^k\right) q_n} - (v - \omega) \\
&> (v - b(v))\frac{G(v)}{\sum_{n=0}^{m} \frac{1}{n+1} \left(\sum_{k=0}^{n} F(v_*)^k\right) q_n} - (v - \omega) \\
&= E[\Pi_1(v, b(v), b(\vec{V}_{-1}), \vec{X}_{-1}, \Sigma)|C_{v_*}^1, X_1 = 1] - E[\Pi_1(v, \omega, b(\vec{V}_{-1}), \vec{X}_{-1}, \Sigma)|C_{v_*}^1, X_1 = 1] \\
&\geq 0, \qquad \textbf{Equation 84.}
\end{aligned}$$

where the last inequality is justified because b is a permissible SBNE. Thus, $\Delta(v) \geq 0$ for $v \leq v_*$. Since Δ is decreasing, for $v < v_*$ we have $\Delta(v) > \Delta(v_*) \geq 0$, completing the proof. □

Heuristically, we assume that any equilibrium strategy for the SFPBO auction should have a jump discontinuity at v_* in the case that $v_* < 1$, as bidders should be willing to bid a higher amount to account for the certainty in using the buy-out price ω. Using **Proposition 23** we can prove this as a fact. Before we do so, we prove a lemma that will be useful in that and subsequent proofs.

Lemma 24. *Define $\Psi : [0, 1] \to [0, 1]$ as $\Psi(v) = G(v) - P(C_v^1|X_1 = 1)$. Then $\Psi(v) < 0$ for $v \in [0, 1)$ and $\Psi(1) = 0$.*

Proof. Let $v \in [0, 1]$. We compute

$$\begin{aligned}
\Psi(v) &= G(v) - P(C_v^1|X_1 = 1) \qquad \textbf{Equation 85.} \\
&= \sum_{n=0}^{m} \frac{1}{(n+1)(1 - F(v))} \left((n+1)F(v)^n(1 - F(v)) - (1 - F(v)^{n+1})\right) q_n \\
&= \sum_{n=0}^{m} \frac{1}{(n+1)(1 - F(v))} j_{n+1}(F(v)) q_n. \qquad \textbf{Equation 86.}
\end{aligned}$$

The result then follows from the facts that $0 \leq F(v) \leq 1$, $F(v) = 1$ if and only if $v = 1$, and **Lemma 22**. □

Now we can prove our jump discontinuity proposition:

Proposition 25. *Assume the conditions of the SFPBO auction, suppose that there exists a permissible SBNE b for this auction with corresponding value v_*, and define Δ as in **Definition 19**. If $v_* < 1$, then $b(v_*) < \omega$.*

Proof. Suppose $v_* < 1$ and $b(v_*) = \omega$ in order to derive a contradiction. Since $b(v_*) = \tilde{b}(v_*)$ and we assume $b(v_*) = \omega$ we have

$$\begin{aligned}
\Delta(v_*) &= (v_* - b(v_*))\frac{G(v_*)}{P(C_{v_*}^1|X_1 = 1)} - (v_* - \omega) \\
&= (v_* - \omega)\left(\frac{G(v_*)}{P(C_{v_*}^1|X_1 = 1)} - 1\right) < 0, \qquad \textbf{Equation 87.}
\end{aligned}$$

where in the last inequality we used the fact that $v_* < 1$ and **Lemma 24**. However, by **Proposition 23**, $\Delta(v_*) \geq 0$, which is a contradiction. Thus, if $v_* < 1$, then $b(v_*) < \omega$. □

If $v_* < 1$, the value v_* of a permissible SBNE b is a tipping point value where a risk-neutral bidder should be indifferent between submitting an offer or just purchasing the item for ω. Since Δ measures the difference in expected surplus between those two options, we thus expect that $\Delta(v_*) = 0$. If $v_* = 1$, then ω is too high for a bidder to gain more expected surplus by submitting the buy-out price of ω than just submitting an offer. We can now prove these two observations.

Proposition 26. *Assume the conditions of the SFPBO auction, suppose that there exists a permissible SBNE* $b : [0,1] \to [0,\omega]$ *for this auction with corresponding value* v_**, and define* Δ *as in* ***Definition 19****. If* $v_* = 1$*, then* $\Delta(v) > 0$ *on* $[0,1)$ *and* $\Delta(v_*) \geq 0$*. If* $v_* < 1$*, then* $\Delta(v_*) = 0$*.*

Proof. If $v_* = 1$, the result directly follows from **Proposition 23**. So suppose $v_* < 1$. By **Proposition 25**, $b(v_*) < \omega$. Let $v \in (v_*, 1]$ be arbitrary. Since b is a permissible SBNE, we have by **Proposition 17**

$$\begin{aligned}
0 &\geq E[\Pi(v, b(v_*), b(\vec{V}_{-1}), X_{-1}, \Sigma)|C^1_{v_*}, X_1 = 1] - E[\Pi(v, b(v), b(\vec{V}_{-1}), X_{-1}, \Sigma)|C^1_{v_*}, X_1 = 1] \\
&= (v - \tilde{b}(v_*))\frac{G(v_*)}{P(C^1_{v_*}|X_1 = 1)} - (v - \omega) \\
&= (v - \tilde{b}(v))\frac{G(v)}{P(C^1_{v}|X_1 = 1)} - (v - \omega) + (v - \tilde{b}(v_*))\frac{G(v_*)}{P(C^1_{v_*}|X_1 = 1)} - (v - \tilde{b}(v))\frac{G(v)}{P(C^1_{v}|X_1 = 1)} \\
&= \Delta(v) + (v - \tilde{b}(v_*))\frac{G(v_*)}{P(C^1_{v_*}|X_1 = 1)} - (v - \tilde{b}(v))\frac{G(v)}{P(C^1_{v}|X_1 = 1)} \\
&= \Delta(v) + \frac{(v - v_*)G(v_*) + \int_0^{v_*} G(y)dy}{P(C^1_{v_*}|X_1 = 1)} - \frac{\int_0^{v} G(y)dy}{P(C^1_{v}|X_1 = 1)}.
\end{aligned} \qquad \textbf{Equation 88.}$$

Since Δ is continuous at v_* and $v_* < 1$, we have

$$\begin{aligned}
\Delta(v_*) &= \lim_{v \to v_*^+} \Delta(v) \\
&\leq \lim_{v \to v_*^+} \frac{\int_0^{v} G(y)dy}{P(C^1_{v}|X_1 = 1)} - \frac{(v - v_*)G(v_*) + \int_0^{v_*} G(y)dy}{P(C^1_{v_*}|X_1 = 1)} \\
&= \frac{\int_0^{v_*} G(y)dy}{P(C^1_{v_*}|X_1 = 1)} - \frac{(v_* - v_*)G(v_*) + \int_0^{v_*} G(y)dy}{P(C^1_{v_*}|X_1 = 1)} \\
&= 0.
\end{aligned} \qquad \textbf{Equation 89.}$$

Thus, $\Delta(v_*) \leq 0$. But by **Proposition 23**, $\Delta(v_*) \geq 0$. Thus $\Delta(v_*) = 0$, as desired. □

We now have completely characterized the permissible SBNE in a SFPBO auction which we summarize in the following theorem.

Theorem 27. *Assume the conditions of the SFPBO auction, suppose that there exists a permissible SBNE* $b : [0,1] \to [0,\omega]$ *for this auction with corresponding value* v_**, and define* Δ *as in* ***Definition 19****. If* $\Delta(v) \geq 0$ *for all* $v \in [0,1]$*, then*

$$b(v) = \begin{cases} 0 & \text{for } v = 0 \\ v - \dfrac{\displaystyle\int_0^v G(y)dy}{G(v)} & \text{for } 0 < v \end{cases}. \qquad \textbf{Equation 90.}$$

If $\Delta(v) < 0$ *for some* $v \in [0,1]$*, then*

$$b(v) = \begin{cases} 0 & \text{for } v = 0 \\ v - \dfrac{\displaystyle\int_0^v G(y)dy}{G(y)} & \text{for } 0 < v \leq v_* \\ \omega & \text{for } v > v_* \end{cases}. \qquad \textbf{Equation 91.}$$

Moreover, $b(v)$ has a jump discontinuity at v_* and v_* is the unique value in $[0,1]$ such that $\Delta(v_*)=0$.

Proof. First suppose $\Delta(v) \geq 0$ for all $v \in [0,1]$. If $v_* < 1$, then by **Proposition 26** $\Delta(v_*) = 0$ and since Δ is decreasing, we have $\Delta(1) < 0$, which contradicts our assumption. Thus, $v_* = 1$ and **Equation 90** follows from **Theorem 18**. Next suppose $\Delta(v) < 0$ for some $v \in [0,1]$. If $v_* = 1$, then by **Proposition 23** $\Delta(v) \geq 0$ for all $v \in [0,1]$, a contradiction. Thus, $v_* < 1$ and **Equation 91** and the comment following it follow from **Theorem 18**, **Proposition 25**, and **Proposition 26**, completing the proof. □

Note that if **Equation 90** holds, the buy-out price ω is higher than any bidder is willing to offer in equilibrium. However, if **Equation 91** holds, not only are some bidders with values $v_* < v$ willing to bid ω, but since there is a jump discontinuity some of them will bid ω even though the bid $\tilde{b}(v)$ is lower than the buy-out price. That is, the buy-out price actually can induce higher bids in equilibrium than the standard first-price sealed-bid auction that does not have a buy-out. Despite this jump discontinuity, we will show in our last section that a seller maximizes revenue by setting a buy-out price that is higher than any bidder is willing to pay in equilibrium.

While **Theorem 27** gives us the exact necessary conditions for a permissible SBNE in the SFPBO auction, we still need to prove this formula gives an actual permissible SBNE. That is, we have proven a uniqueness statement in the previous theorem, but we still need to prove existence. Proving that the formulas **Equation 90** and **Equation 91** actually give permissible SBNEs the content of the proof of the next theorem.

Theorem 28. *Assume the conditions of the SFPBO auction, define Δ as in **Definition 19**, and define v_* as*

$$v_* = \inf(\{v \in [0,1] : \Delta(v) < 0\} \cup \{1\}) \qquad \text{Equation 92.}$$

Then the function $b : [0,1] \to [0,1]$ given by

$$b(v) = \begin{cases} 0 & \text{for } v = 0 \\ v - \dfrac{\int_0^v G(y)dy}{G(v)} & \text{for } 0 < v \leq v_* \\ \omega & \text{for } v > v_* \end{cases} . \qquad \text{Equation 93.}$$

is the unique permissible SBNE for the SFPBO auction.

Proof. The uniqueness part of this statement was proven in **Theorem 27**, thus it only remains to prove that b as defined in **Equation 93** yields a permissible SBNE. Due to the symmetry of $\vec{V}$ and $\vec{X}$ as well as the fact that the quantities $P(C^i_{v_*}|X_i = 1)$ (**Proposition 12**) and $E_i[v, \hat{b}, b(\vec{V}_{-i}), \vec{X}_{-i}, \Sigma | C^i_{v_*}, X_i = 1]$ (**Proposition 17**) are independent of i, it suffices to prove the result for $i = 1$. First we record some facts about Δ and v_*. Note that since $\Delta(0) = \omega > 0$ and Δ is continuous, we must have $v_* > 0$. Also by continuity and the fact that Δ is decreasing, $\Delta(v_*) \geq 0$ and thus $\Delta(v) > 0$ for $v < v_*$. Finally, if $v_* < 1$, again by continuity $\Delta(v_*) = 0$ and since Δ is decreasing if $v > v_*$, then $\Delta(v) < 0$.

Next we will show b is permissible. Note that b is differentiable on $(0, v_*)$ and continuous on $(0, v_*]$ by the Fundamental Theorem of Calculus. To show that b is also continuous at 0 we can employ a Squeeze Theorem argument. To that end, first note that $G'(y) > 0$ since $F'(y) = f(y) > 0$ for all $y \in (0,1)$, so we have

$$b(v) = \int_0^v yG'(y)dy > 0 \text{ for } v \in (0, v_*). \qquad \text{Equation 94.}$$

Also, for $v \in (0, v_*)$, $b(v) < v$. Thus, by the Squeeze Theorem, $\lim_{v \to 0^+} b(v) = 0 = b(0)$, proving continuity at 0. To show that b is increasing on $[0, v_*]$, we will show $b'(v) > 0$ for $v \in (0, v_*)$. Again noting that $G'(v) > 0$ for $v \in (0, v_*)$, we have

$$b'(v) = 1 - \frac{G(v)^2 - G'(v)\int_0^v G(y)dy}{G(v)^2} = \frac{G'(v)\int_0^v G(y)dy}{G(v)^2} > 0. \qquad \text{Equation 95.}$$

To prove that b is nondecreasing on $[0,1]$ and $v_* = \inf(\{v \in [0,1] : b(v) = \omega\} \cup \{1\})$, it suffices to show that $b(v) < \omega$ for $v < v_*$. Suppose $b(v) = \omega$ for some $v < v_*$ to derive a contradiction. Since $v < v_* \le 1$, we have $G(v) - P(C_v^1|X_1 = 1) < 0$ by **Lemma 24**. So we have

$$0 < \Delta(v) = (v - b(v))\frac{G(v)}{P(C_v^1|X_1 = 1)} - (v - \omega) = (v - \omega)\left(\frac{G(v)}{P(C_v^1|X_1 = 1)} - 1\right) < 0, \quad \textbf{Equation 96.}$$

giving us our desired contradiction. Thus, $b(v) < \omega$ for all $v < v_*$ and we have shown that b is permissible.

Next we show that b is a permissible SBNE. Note that for any $v \in [0,1]$, by **Proposition 17**, if we have $\hat{b} \in (b(v_*), \omega)$, then

$$\begin{aligned} E\left[\Pi_1(v, \hat{b}, b(\vec{V}_{-1}), X_{-1}, \Sigma)\,\middle|C_{v_*}^1, X_1 = 1\right] &= \frac{(v - \hat{b})G(v_*)}{P(C_{v_*}^1|X_1 = 1)} \\ &< \frac{(v - b(v_*))G(v_*)}{P(C_{v_*}^1|X_1 = 1)} \\ &= E\left[\Pi_1(v, b(v_*), b(\vec{V}_{-1}), X_{-1}, \Sigma)\,\middle|C_{v_*}^1, X_1 = 1\right]. \end{aligned} \quad \textbf{Equation 97.}$$

Since $b(0) = 0$, to prove b is a permissible SBNE it suffices to show that for all $v \in [0,1]$,

$$E\left[\Pi_1(v, b(v), b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C_{v_*}^1, X_1 = 1\right] = \sup_{\hat{b} \in [b(0), b(v_*)] \cup \{\omega\}} E\left[\Pi_1(v, \hat{b}, b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C_{v_*}^1, X_1 = 1\right]$$

Equation 98.

To that end, for any $x \in [0, v_*) \cup \{v|v = v_* \text{ and } b(v) < \omega\}$, we have by **Proposition 17**

$$\begin{aligned} E\left[\Pi_1(v, b(x), b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C_{v_*}^1, X_1 = 1\right] &= \frac{(v - b(x))G(x)}{P(C_{v_*}^1, X_1 = 1)} \\ &= \frac{1}{P(C_{v_*}^1, X_1 = 1)}\left((v - x)G(x) + \int_0^x G(y)dy\right) \end{aligned} \quad \textbf{Equation 99.}$$

Taking the derivative of the above with respect to x (for $x \in (0, v_*)$) yields

$$\frac{d}{dx}E\left[\Pi_1(v, b(x), b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C_{v_*}^1, X_1 = 1\right] = \frac{1}{P(C_{v_*}^1, X_1 = 1)}(v - x)G'(x). \quad \textbf{Equation 100.}$$

Since $G'(x) = \sum_{n=1}^m nF(x)^{n-1}f(x)q_n > 0$ for all x in $(0,1)$, the above derivative is positive for $x < v$ and negative for $x > v$. Thus, in the case where $v < v_*$, we have a global maximum for our conditional expected value at $b(v)$ over the set $[b(0), b(v_*)) \cup \{b(v)|v = v_* \text{ and } b(v) < \omega\}$. In the case where $v \ge v_*$, then we have

$$\sup_{\hat{b} \in [b(0), b(v_*)) \cup \{b(v)|v = v_*, b(v) < \omega\}} E\left[\Pi_1(v, \hat{b}, b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C_{v_*}^1, X_1 = 1]\right] \le \frac{(v - b(v_*))G(v_*)}{P(C_{v*}^1, X_1 = 1)}. \quad \textbf{Equation 101.}$$

Putting these two facts in one expression, we have

$$\begin{aligned} &\sup_{\hat{b} \in [b(0), b(v_*)) \cup \{b(v)|v = v_*, b(v) < \omega\}} E\left[\Pi_1(v, \hat{b}, b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C_{v_*}^1, X_1 = 1\right] \\ &= \begin{cases} E\left[\Pi_1(v, b(v), b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C_{v_*}^1, X_1 = 1\right] & \text{if } v < v_* \\ \dfrac{(v - b(v_*))G(v_*)}{P(C_{v*}^1, X_1 = 1)} & \text{if } v \ge v_*. \end{cases} \end{aligned} \quad \textbf{Equation 102.}$$

We next aim to show that $E\left[\Pi(v, b(v), b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C_{v_*}^1, X_1 = 1\right] > (v - \omega) = E\left[\Pi(v, \omega, b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C_{v_*}^1, X_1 = 1\right]$ for $v < v_*$. To that end, define a function $\tilde{\Delta}(v)$ given by

$$\tilde{\Delta}(v) = \frac{(v - \tilde{b}(v))G(v)}{P(C_{v_*}^1|X_1 = 1)} - (v - \omega) = \frac{1}{P(C_{v_*}^1|X_1 = 1)}\int_0^v G(y)dy - v + \omega. \quad \textbf{Equation 103.}$$

Note that for $v < v_*$

$$\tilde{\Delta}'(v) = \frac{1}{P(C^1_{v_*}|X_1 = 1)}(G(v) - P(C^1_{v_*}|X_1 = 1)) < 0 \qquad \text{Equation 104.}$$

where the last inequality is justified by **Lemma 24**. So $\tilde{\Delta}$ is decreasing on $[0, v_*]$ and $\tilde{\Delta}(v_*) = \Delta(v_*) \geq 0$ (due to the definition of v_* and the fact that b is increasing and continuous on $[0, v_*]$) which implies $\tilde{\Delta}(v) > 0$ for $v < v_*$. Thus we indeed have

$$E\left[\Pi_1(v, b(v), b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C^1_{v_*}, X_1 = 1\right] - E\left[\Pi_1(v, \omega, b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C^1_{v_*}, X_1 = 1\right] = \tilde{\Delta}(v) > 0 \qquad \text{Equation 105.}$$

for $v < v_*$. Thus, by **Equation 102**, **Equation 98** holds when $v < v_*$. Next we consider the case $v = v_*$. If $b(v_*) < \omega$, then using **Equation 102** and the fact that $\Delta(v_*) \geq 0$ we have

$$\begin{aligned} E\left[\Pi_1(v_*, b(v_*), b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C^1_{v_*}, X_1 = 1\right] =& \frac{(v_* - b(v_*))G(v_*)}{P(C^1_{v*}, X_1 = 1)} \\ \geq& (v - \omega) \\ =& E\left[\Pi_1(v_*, \omega, b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C^1_{v_*}, X_1 = 1\right], \end{aligned} \qquad \text{Equation 106.}$$

which by **Equation 102** proves **Equation 98** when $v = v_*$ and $b(v_*) < \omega$. Next we consider the case that $v = v_*$ and $b(v_*) = \omega$. Note that $\Delta(v) > 0$ for $v < v_*$ and $\Delta(v_*) \leq 0$ by the definition of v_* and the fact that Δ is increasing and continuous on $[0, v_*]$. So, by the same reasoning found in **Proposition 25** we can see that both $v_* = 1$ and $\Delta(v_*) = 0$. Thus, if $b(v_*) = \omega$, then

$$E\left[\Pi_1(v_*, b(v_*), b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C^1_{v_*}, X_1 = 1\right] = v_* - \omega = \frac{(v_* - b(v_*))G(v_*)}{P(C^1_{v_*}, X_1 = 1)}. \qquad \text{Equation 107.}$$

So again by **Equation 102** and the fact $b(v_*) = \omega$, the above proves **Equation 98** when $v = v_*$ and $b(v_*) = \omega$. Finally, if $v > v_*$, we consider the function $\tilde{E} : [0, 1] \to [0, \infty)$ defined by $\tilde{E}(x) = (v - \tilde{b}(x))G(x)/P(C^1_{v_*}|X_1 = 1) = ((v - x)G(x) + \int_0^x G(y)dy)/P(C^1_{v_*}|X_1 = 1)$. Note that (similar to above) $\tilde{E}'(x) = (v - x)G'(x)$ and so $\tilde{E}(x)$ takes a global maximum value at $x = v$. Using that fact, **Equation 102**, and the fact that $\Delta(v) < 0$ (as $v > v_*$), we have

$$\begin{aligned} &\sup_{\hat{b} \in [b(0), b(v_*)) \cup \{b(v)|v = v_*, b(v) < \omega\}} E\left[\Pi_1(v, \hat{b}, b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C^1_{v_*}, X_1 = 1\right] \\ &= (v - b(v_*))G(v_*) \\ &= (v - \tilde{b}(v_*))G(v_*) \\ &= \tilde{E}(v_*) \\ &< \tilde{E}(v) \\ &= (v - \tilde{b}(v))G(v) < (v - \omega) \\ &= E\left[\Pi_1(v, b(v), b(\vec{V}_{-1}), X_{-1}, \Sigma)\middle| C^1_{v_*}, X_1 = 1]\right]. \end{aligned} \qquad \text{Equation 108.}$$

Thus, **Equation 98** holds when $v > v_*$ and so **Equation 98** holds for all $v \in [0, 1]$, completing the proof. □

A MODEL WITH SIMULTANEOUS BIDDING

One of the hallmarks of the SFPBO auction was the serial nature of the auction. The reason we considered bidders bidding in serial was to reflect how bidders bid on the Best Offer auction on eBay. In particular, if a bidder on eBay submits the BIN price, then that bidder will win the auction with certainty. However, often in auction theory auction models are created bidders submit their bids simultaneously and any resulting ties are broken randomly. For the second-price with buy-out auction considered in the work of Reynolds and Wooders this latter approach is used.[3] Here we consider how, if at all, the permissible SBNE strategies for our SFPBO auction would change if we took this alternate approach. We will call the resulting model the First-Price Buy Out (FPBO) model and define it below:

Definition 29. *The **conditions of the First Price Buy Out (FPBO) auction** are conditions (i) through (xi) of the definition of the SFPBO auction (**Definition 9**) with the added condition*

(xii) all bidders bid simultaneously.

Since the bidders in the FPBO bid simultaneously, all bidders have the opportunity to bid. Thus, we modify our definition of the SBNE for this model:

Definition 30. *Assume the conditions of the FPBO auction. We say that* $b : [0,1] \to [0,\omega]$ *is a* permissible symmetric Bayesian Nash equilibrium bidding strategy for the FPBO auction *(or just* permissible SBNE*) if* b *is permissible and for all* $i \in \{1,\ldots,m+1\}$

$$\sup_{\hat{b}\in[0,\omega]} E\left[\Pi_i(v_i,\hat{b},b(\vec{V}_{-i}),\vec{X}_{-i},\Sigma)\middle| X_i=1\right] = E\left[\Pi_i(v_i,b(v_i),b(\vec{V}_{-i}),\vec{X}_{-i},\Sigma)\middle| X_i=1\right]. \quad \text{Equation 109.}$$

We will derive for a SBNE for this FPBO auction. Since the derivation is similar to the SBNE for the SFPBO auction, we will omit many of the details of the derivation, focusing only on important differences.

For the SBNE for the FPBO auction, we no longer condition on the event $C^i_{v_*}$, which makes deriving the SBNE easier, but also means that a bidder who submits the maximum bid ω no longer wins the item with certainty. In fact, given a particular threshold v_*, if bidder i bids ω, then bidder i will win if for all $j \in \{1,\ldots,m+1\}$, if $\Sigma(j) < \Sigma(i)$ then $X_j = 0$ or $V_j \le v_*$. That is, bidder i wins in the case of the event $C^i_{v_*}$. More precisely, we have an analogue to **Proposition 17**:

Proposition 31. *Assume the conditions of the FPBO auction. Let* $b : [0,1] \to [0,\omega]$ *be a permissible bidding function for the FPBO auction with the cut-off value of* v_**, and suppose that* $b(0) = 0$*. Let* b^{-1} *be the inverse of* $b|_{[0,v_*]}$ *and fix* $i \in \{1,\ldots,m+1\}$*. If* $v \in [0,1]$ *and* $\hat{b} \in [0,\omega]$*, then*

$$E\Big[\Pi_i\big(v,\hat{b},b(\vec{V}_{-i}),\vec{X}_{-i},\Sigma\big)\Big|X_i=1\Big] = \begin{cases} (v-\hat{b})G(b^{-1}(\hat{b})) & \text{if } \hat{b} \le b(v_*) \text{ and } \hat{b} < \omega \\ (v-\hat{b})G(v_*) & \text{if } b(v_*) < \hat{b} < \omega \\ (v-\omega)P(C^1_{v_*}|X_1=1) & \text{if } \hat{b} = \omega. \end{cases} \quad \text{Equation 110.}$$

In particular, $E\left[\Pi_i\left(v,\hat{b},b(\vec{V}_{-i}),\vec{X}_{-i},\Sigma\right)\middle| X_i=1\right]$ *is independent of* i*.*

We omit a proof of the above as the proposition can be proven analogously to the proof of **Proposition 17**. From **Proposition 31**, one can prove an analogue to **Theorem 18**:

Theorem 32. *Assume the conditions of the FPBO auction. Let* $b : [0,1] \to [0,\omega]$ *be a permissible SBNE for the FPBO auction. Then, if* $v \in [0,v_*]$

$$b(v) = \tilde{b}(v) = \begin{cases} v - \dfrac{\int_0^v G(y)dy}{G(v)} & \text{if } 0 < v \le v_* \\ 0 & \text{if } v = 0. \end{cases} \quad \text{Equation 111.}$$

Again, we omit the proof as it is analogous to the proofs of **Proposition 13**, **Proposition 14**, and **Theorem 18**. Note that **Theorem 32** implies that any SBNE to the FPBO auction must be the same as as the unique SBNE to the SFPBO auction before the cut-off value v_*. But must the cut-off value be the same? To answer this question, we need a new version of our difference function Δ:

Definition 33. *Assume conditions (i) through (xi) of the SFPBO auction. Define the function* $\overline{\Delta} : [0,1] \to \mathbb{R}$ *by*

$$\overline{\Delta}(v) := (v-\tilde{b}(v))G(v) - (v-\omega)P\left(C^1_v\middle| X_1=1\right) = \int_0^v G(y)dy - (v-\omega)P\left(C^1_v\middle| X_1=1\right) \quad \text{Equation 112.}$$

In light of **Proposition 31** and **Theorem 32**, $\overline{\Delta}$ measures the difference in expected value of bidding using the SBNE without a cutoff $\tilde{b}$ and using the buy out price ω. In the next proposition we state the obvious fact that Δ and $\overline{\Delta}$ are related, and thus many of the properties we proved about Δ carry over to $\overline{\Delta}$:

Proposition 34. *Assume conditions (i) through (xi) of the SFPBO auction. Then*

$$\overline{\Delta}(v) = P\left(C_v^1 \middle| X_1 = 1\right)\Delta(v). \qquad \text{Equation 113.}$$

Furthermore, $\overline{\Delta}(0) > 0$ *and we have* $\overline{\Delta}(v) = 0 \iff \Delta(v) = 0$. *Still furthermore, if* b *is a permissible SBNE for the FPBO auction with associated cut-off value* v_*, *then* $\overline{\Delta}(v) > 0$ *for* $v \in (0, v_*)$ *and* $\overline{\Delta}(v_*) \geq 0$. *If* $v_* < 1$ *then* $b(v_*) < \omega$ *and* $\overline{\Delta}(v_*) = 0$.

Proof. **Equation 113** follows directly from the definitions of Δ and $\overline{\Delta}$. Since $P\left(C_v^1 \middle| X_1 = 1\right) > 0$, $\overline{\Delta}(v) = 0 \iff \Delta(v) = 0$ holds by **Equation 113**. $\overline{\Delta}(0) > 0$ follows from **Proposition 20** and **Equation 113**. The properties $\overline{\Delta}(v) > 0$ for $v \in (0, v_*)$ and $\overline{\Delta}(v_*) \geq 0$ can be proven an analogously to **Proposition 23**. If $v_* < 1$, then $b(v_*) < \omega$ follows from a proof analogous to that of **Proposition 25**. Finally $\overline{\Delta}(v_*) = 0$ follows by a proof analogous to the proof of **Proposition 26**. ☐

We can now state the main theorem of this section.

Theorem 35. *Assume the conditions of the FPBO auction, suppose that there exists a SBNE* $b : [0,1] \to [0,\omega]$ *for this auction with corresponding value* v_*. *If* $\overline{\Delta}(v) \geq 0$ *for all* $v \in [0,1]$, *then*

$$b(v) = \begin{cases} 0 & \text{for } v = 0 \\ v - \dfrac{\int_0^v G(y)dy}{G(v)} & \text{for } 0 < v \end{cases}. \qquad \text{Equation 114.}$$

If $\overline{\Delta}(v) < 0$ *for some* $v \in [0,1]$, *then*

$$b(v) = \begin{cases} 0 & \text{for } v = 0 \\ v - \dfrac{\int_0^v G(y)dy}{G(y)} & \text{for } 0 < v \leq v_* \\ \omega & \text{for } v > v_* \end{cases}. \qquad \text{Equation 115.}$$

Moreover, v_* *is the unique value in* $[0,1]$ *such that* $\overline{\Delta}(v_*) = 0$ *and* $b(v)$ *is equal to the unique SBNE for the SFPBO auction.*

Proof. First suppose $\overline{\Delta}(v) \geq 0$ for all $v \in [0,1]$. If $v_* < 1$, then by **Proposition 34** $\overline{\Delta}(v_*) = 0$. Again by **Proposition 34** we thus know $\Delta(v_*) = 0$ and since Δ is decreasing, we have $\Delta(1) < 0$. So $\overline{\Delta}(1) = P\left(C_1^1 \middle| X_1 = 1\right)\Delta(1) < 0$, which contradicts our assumption. Thus, $v_* = 1$ and **Equation 114** follows from **Theorem 32**. Next suppose $\overline{\Delta}(v) < 0$ for some $v \in [0,1]$. If $v_* = 1$, then by **Proposition 34** $\overline{\Delta}(v) \geq 0$ for all $v \in [0,1]$, a contradiction. Thus, $v_* < 1$ and **Equation 115** and the first assertion following it follow from **Theorem 32** and **Proposition 34**. The last assertion follows from **Equation 113** and **Theorem 27**. ☐

Heuristically, this result is surprising. In the SFPBO auction, a bidder not only knows that she will win with certainly if she bids ω, but that whatever bidders bid before her must have values less than the cut-off value. However, in equilibrium she behaves as she would in the FPBO auction. That is, in equilibrium in the SFPBO auction she behaves as if she did not have this extra information. As we noted (in more detail) in the comment following **Proposition 17**, this equivalency is due to the fact that if a bidder has a value under the cut-off value of v_*, she can only win if all bidders also have values under the cut-off value.

SELLER STRATEGY IN SETTING THE BUY-OUT PRICE

In the SFPBO/FPBO auction model, the seller seems to do himself a disservice by putting a ceiling (the buy-out price ω) on the highest possible bid. However, playing devil's advocate, one could dispute this heuristic point. Indeed, such a contrarian could point to the fact that the unique permissible SBNE can have a jump discontinuity, and thus in certain situations buyers could bid more in a SFPBO/FPBO auction than a standard first-price sealed-bid auction. Perhaps a seller could set a buy-out price ω such that his expected revenue in this type of auction is the same or greater than the expected revenue in a straight first-price sealed-bid auction with no minimum bid (that is, an auction that does not have a ceiling on a high bid). Furthermore, because in the SFPBO/FPBO model the seller cannot set an open reserve price other than 0, setting ω higher than any bidder is willing to pay is not an optimal auction.[7] That said, we will now contradict that contrarian point of view and prove that setting ω higher than any bidder is willing to pay does generate the maximal expected revenue to the seller.

Recall that cut-off value v_* is implicitly determined by the equation $\Delta(v) = 0$ (as long as $v_* < 1$) and Δ depends on ω. Solving the equation $\Delta(v_*) = 0$ for ω yields

$$\omega = v_* - \frac{\int_0^{v_*} G(y)dy}{P(C^1_{v_*}|X_1 = 1)}. \qquad \text{Equation 116.}$$

If $v_* = 1$, then the bidders always use the strategy $\tilde{b}$ and never submit the buy-out price of $\omega = 1 - \frac{\int_0^1 G(y)dy}{P(C^1_1|X_1=1)}$. Setting ω higher than $1 - \frac{\int_0^1 G(y)dy}{P(C^1_1|X_1=1)}$ would have the same effect of bidders never submitting the buy-out price as a bid, so we can find an optimal cut-off value v_* and use it to determine a corresponding optimal ω.

Theorem 36. *Assume the conditions (i) through (xi) of the SFPBO auction. For any $v_* \in [0, 1]$, let $b_{v_*}(v)$ be the SBNE of the SFPBO auction as defined in **Theorem 28** with the corresponding cut-off value of v_*. Let $R_{v_*} = b_{v_*}(\tilde{Y})$ be the revenue generated by the SFPBO auction with $\omega = v_* - \frac{\int_0^{v_*} G(y)dy}{P(C^1_{v_*}|X_1=1)}$. The quantity $E[R_{v_*}]$ is maximized at $v_* = 1$.*

Proof. First we derive a formula for the cdf $\tilde{G}(y)$ of $\tilde{Y} = \max_{1\le j\le m+1} X_j V_j$. We have

$$\tilde{G}(y) = P(Y \le y) = \sum_{n=0}^{m+1} P(Y \le y|N = n)P(N = n) = \sum_{n=0}^{m+1} F(y)^n \tilde{q}_n, \qquad \text{Equation 117.}$$

as $\tilde{q}_n = P(N = n)$ by definition of $\tilde{q}_n$. As our formula for $E[R_{v_*}]$ will involve both $\tilde{q}_n$'s (from $\tilde{G}$) and q_n's (from G), we will want a relationship between the two quantities. To that end, for any $n \in \{2, \dots, m + 1\}$ we have

$$\begin{aligned}
q_n &= P(N = n + 1|X_1 = 1)\\
&= \frac{1}{P(X_1 = 1)} P(N = n + 1, X_1 = 1)\\
&= \frac{1}{P(X_1 = 1)} \sum_{\vec{\beta}\in\{0,1\}^m, \sum_{i=1}^m \beta_i = n} P(X_1 = 1, X_2 = \beta_1, X_3 = \beta_2, \dots, X_{m+1} = \beta_m)\\
&= \frac{1}{P(X_1 = 1)} \sum_{\vec{\beta}\in\{0,1\}^m, \sum_{i=1}^m \beta_i = n} P(X_1 = 1, X_2 = 1, \dots, X_{n+1} = 1, X_{n+2} = 0, \dots, X_{m+1} = 0)\\
&= \frac{1}{P(X_1 = 1)} \binom{m}{n} P(X_1 = 1, X_2 = 1, \dots, X_{n+1} = 1, X_{n+2} = 0, \dots, X_{m+1} = 0)\\
&= \frac{1}{P(X_1 = 1)} \frac{\binom{m}{n}}{\binom{m+1}{n+1}} \tilde{q}_{n+1}\\
&= \frac{1}{P(X_1 = 1)} \frac{n+1}{m+1} \tilde{q}_{n+1}.
\end{aligned} \qquad \text{Equation 118.}$$

Now we can start to compute $E[R_{v_*}]$:

$$\begin{aligned} E[R_{v_*}] &= E[b_{v_*}(Y)] \\ &= \int_0^1 b_{v_*}(y)\tilde{G}'(y)dy \\ &= \int_0^{v_*} \tilde{b}(y)\tilde{G}'(y)dy + \int_{v_*}^1 \omega\tilde{G}'(y)dy \\ &= \int_0^{v_*} \tilde{b}(y)\tilde{G}'(y)dy + \omega(1-\tilde{G}(v_*)) \\ &= \int_0^{v_*} \left(y - \frac{\int_0^y G(x)dx}{G(y)}\right)\tilde{G}'(y)dy + \left(v_* - \frac{\int_0^{v_*} G(y)dy}{P(C^1_{v_*}|X_1=1)}\right)(1-\tilde{G}(v_*)). \end{aligned} \quad \textbf{Equation 119.}$$

In the last line of **Equation 119**, the quantity $\frac{\tilde{G}'(y)}{G(y)}$ appears. To move forward, we first simplify that quotient using **Equation 118**:

$$\begin{aligned} \frac{\tilde{G}'(y)}{G(y)} &= \frac{\sum_{n=0}^m (n+1)F(y)^n f(y)\tilde{q}_{n+1}}{\sum_{n=0}^m F(y)^n q_n} \\ &= (m+1)P(X_1=1)f(y)\frac{\sum_{n=0}^m F(y)^n q_n}{\sum_{n=0}^m F(y)^n q_n} \\ &= (m+1)P(X_1=1)f(y). \end{aligned} \quad \textbf{Equation 120.}$$

Plugging **Equation 120** into **Equation 119** and integrating by parts in the second integral of the second line yields

$$\begin{aligned} E[R_{v_*}] &= \int_0^{v_*} \left(y - \frac{\int_0^y G(x)dx}{G(y)}\right)\tilde{G}'(y)dy + \left(v_* - \frac{\int_0^{v_*} G(y)dy}{P(C^1_{v_*}|X_1=1)}\right)(1-\tilde{G}(v_*)) \\ &= \int_0^{v_*} y\tilde{G}'(y)dy - (m+1)P(X_1=1)\int_0^{v_*}\left(\int_0^y G(x)dx\right)f(y)dy + \left(v_* - \frac{\int_0^{v_*} G(y)dy}{P(C^1_{v_*}|X_1=1)}\right)(1-\tilde{G}(v_*)) \\ &= \int_0^{v_*} y\tilde{G}'(y)dy - (m+1)P(X_1=1)\left(\left(F(y)\int_0^y G(x)dx\right)_{y=0}^{y=v_*} - \int_0^{v_*} F(y)G(y)dy\right) + \\ &\quad \left(v_* - \frac{\int_0^{v_*} G(y)dy}{P(C^1_{v_*}|X_1=1)}\right)(1-\tilde{G}(v_*)) \\ &= \int_0^{v_*} y\tilde{G}'(y)dy - (m+1)P(X_1=1)\left(\left(F(v_*)\int_0^{v_*} G(y)dy\right) - \int_0^{v_*} F(y)G(y)dy\right) + \\ &\quad v_*(1-\tilde{G}(v_*)) - \frac{1-\tilde{G}(v_*)}{P(C^1_{v_*}|X_1=1)}\int_0^{v_*} G(y)dy \end{aligned} \quad \textbf{Equation 121.}$$

To push the calculation of $E[R_{v_*}]$ forward, we next we simplify the quotient $\frac{1-\tilde{G}(v_*)}{P(C^1_{v_*}|X_1=1)}$. If $v_* = 1$, then the quotient is 0.

If $v_* < 1$ we have:

$$\begin{aligned}
\frac{1-\tilde{G}(v_*)}{P(C^1_{v_*}|X_1=1)} &= \frac{\left(1-\sum_{n=0}^{m+1} F(v_*)^n\tilde{q}_n\right)(1-F(v_*))}{\sum_{n=0}^{m}\frac{1}{n+1}(1-F(v_*)^{n+1})q_n} \\
&= \frac{\left(1-\tilde{q}_0-\sum_{n=0}^{m} F(v_*)^{n+1}\tilde{q}_{n+1}\right)(1-F(v_*))}{\sum_{n=0}^{m}\frac{1}{n+1}(1-F(v)^{n+1})q_n} \\
&= \frac{\left(\sum_{n=1}^{m+1}\tilde{q}_n-\sum_{n=0}^{m} F(v_*)^{n+1}\tilde{q}_{n+1}\right)(1-F(v_*))}{\sum_{n=0}^{m}\frac{1}{n+1}(1-F(v)^{n+1})q_n} \\
&= \frac{\left(\sum_{n=0}^{m}(1-F(v_*)^{n+1})\tilde{q}_{n+1}\right)(1-F(v_*))}{\sum_{n=0}^{m}\frac{1}{n+1}(1-F(v)^{n+1})q_n} \\
&= \frac{(m+1)P(X_1=1)\left(\sum_{n=0}^{m}\frac{1}{n+1}(1-F(v_*)^{n+1})q_n\right)(1-F(v_*))}{\sum_{n=0}^{m}\frac{1}{n+1}(1-F(v)^{n+1})q_n} \\
&= (m+1)P(X_1=1)(1-F(v_*)). \qquad \textbf{Equation 122.}
\end{aligned}$$

Since $(m+1)P(X_1=1)(1-F(1))=0$, we have $\frac{1-\tilde{G}(v_*)}{P(C^1_{v_*}|X_1=1)}=(m+1)P(X_1=1)(1-F(v_*))$ for all v_*. Thus, we can continue **Equation 121**:

$$\begin{aligned}
E[R_{v_*}] = &\int_0^{v_*} y\tilde{G}'(y)dy-(m+1)P(X_1=1)\left(F(v_*)\int_0^{v_*}G(y)dy-\int_0^{v_*}F(y)G(y)dy\right)+ \\
&v_*(1-\tilde{G}(v_*))-\frac{1-\tilde{G}(v_*)}{P(C^1_{v_*}|X_1=1)}\int_0^{v_*}G(y)dy \\
&=\int_0^{v_*} y\tilde{G}'(y)dy+v_*(1-\tilde{G}(v_*))-(m+1)P(X_1=1)\left(\int_0^{v_*}G(y)dy-\int_0^{v_*}F(y)G(y)dy\right). \qquad \textbf{Equation 123.}
\end{aligned}$$

Taking the derivative of the above yields

$$\begin{aligned}
\frac{1}{(m+1)P(X_1=1)}\frac{d}{dv_*}E[R_{v_*}] &= \frac{1}{(m+1)P(X_1=1)}(v_*\tilde{G}'(v_*)+(1-\tilde{G}(v_*))-v_*\tilde{G}'(v_*))-(1-F(v_*))G(v_*) \\
&= \frac{1-\tilde{G}(v_*)}{(m+1)P(X_1=1)}-(1-F(v_*))G(v_*) \\
&= (1-F(v_*))(P(C^1_{v_*}|X_1=1)-G(v_*)) \\
&> 0 \qquad \textbf{Equation 124.}
\end{aligned}$$

for $v_* \in (0,1)$. We used **Equation 122** in the last equality and **Lemma 24** in the last inequality. Thus, $E[R_{v_*}]$ is increasing in v_* and thus must be maximized at the right endpoint $v_*=1$, completing the proof. □

By **Theorem 36**, a seller should set ω high enough so that $v_*=1$. By setting $v_*=1$ in **Equation 116**, any ω bigger than or equal to $1-\frac{\int_0^1 G(y)dy}{P(C^1_1|X_1=1)}=1-\int_0^1 G(y)dy$ would do that job. In this case, the seller has set ω high enough so that in equilibrium no bidder (except possibly a bidder with $v=1$ in the case $\omega=1-\int_0^1 G(y)dy$) will ever use the buy-out price ω, making the bidders behave in equilibrium in both the SFPBO and FPBO auctions as they would in a first-price auction with no reserve price and without a buy-out price.

SUMMARY OF FINDINGS AND FUTURE WORK

Here we take a heuristic account of our results. In the Secret Threshold (ST) game, we assume risk-neutral, independent, and symmetric bidders act in serial, either submit one take-it-or-leave-it offer to the seller or submit a buy-out price, and know the previous number of failed bids. The seller accepts any bid that exceeds some secret threshold (or secret reserve) price as well as the buy-out price. We derived optimal bids for this game that are based on the bidder's value for the object as well as the number of previous failed bids (**Theorem 4** and **Theorem 5**). We also show that given a fixed value and assuming bidders bid optimally, the expected bidder surplus decreases in the number of previous failed bids (**Theorem 8**). Thus, it is better to be an early bidder rather than a late bidder in the ST game, despite the fact that late bidders have more information than early bidders.

In the Serial First-Price Buy-Out (SFPBO) auction, we again assume risk-neutral, independent, and symmetric bidders act in serial and submit one offer to the seller that is at or below the seller's buy-out price. However, in this auction the seller does not have a secret threshold and will instead select the highest bid among the bidders. The auction immediately ends when one bidder submits the buy-out price for her with the buy-out bidder winning the auction and paying the buy-out price. Furthermore, at the time of bidding, bidders do not know how many previous failed bids have been made or how many bids will be made in the future, but do know that any previous bid must have been lower than the buy-out price. We found an equilibrium cut-off bidding strategy for this auction that is unique up to some reasonable conditions (**Theorem 28**). Furthermore, we found that if we changed the SFPBO auction to an auction where bidders acted simultaneously and ties were broken randomly (the First-Price Buy-Out (FPBO) auction), then any reasonable equilibrium cut-off bidding strategy for the FPBO auction must be the same strategy as the equilibrium strategy for the SFPBO auction (**Theorem 35**). This last result is surprising: bidders in the SFPBO auction have more information than bidders in the FPBO auction and yet they behave the same in equilibrium. Lastly, we considered how the seller could optimally set the buy-out price in either the SFPBO or FPBO auctions, and found that the seller should set the buy-out price higher than any bidder in equilibrium would consider submitting (**Theorem 36**). That is, the seller should set a buy-out price so high that the auction is effectively just a standard first-price sealed-bid auction with no reserve price.

One obvious question stemming from this last result is why sellers would choose to sell their goods using a Best Offer auction if the SFPBO and FPBO auctions models offer no more expected revenue than a standard first-price sealed-bid auction with no reserve, an auction that is not typically an optimal auction. An equally obvious possible explanation is that bidders are actually risk-adverse and not risk-neutral, and perhaps the SFPBO/FPBO auctions actually can generate more expected revenue than the first-price auction when bidders are risk-adverse. The risk-adverse assumption is made in Reynolds and Wooders and they show that revenue can be increased with buy-out prices in second-price auction,[3] and thus risk aversion would be a natural assumption to include in future work on deriving equilibrium strategies for first-price with buy-out auctions. In fact, one could then compare the revenue to the second-price with buy-out auction modeled by Reynolds and Wooders with revenue to the first-price with buy-out. In light of our result that the SFPBO and FPBO models have the same equilibria, it would make sense to use the easier FPBO model, particularly since random tie-breaking for bidders using the buy-out price is the assumption found in Reynolds and Wooders work. Perhaps this future work could illuminate why the Buy-It-Now with Best Offer buying format is a popular way for sellers to sell goods on eBay.

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PRESS SUMMARY

Have you ever gone on eBay and wondered how much you should bid? This can be a tricky question, especially for a "Buy-It-Now or Best Offer" auction. Should you buy-it-now or make an offer? If you make an offer, how much should you offer? You want to make an offer that is large enough to have a good chance to win the item for but not too large since you want to get a good deal. In this paper we theoretically answer this question. Under certain assumptions, we mathematically determine when to buy-it-now, when to make an offer, and how much that offer should be.