The Structure of Digroups

Catherine Crompton
Department of Mathematics
Agnes Scott College
141 E. College Avenue
Decatur, Georgia 30030 USA

Linda Scalici
Department of Mathematics
Bucknell University
Moore Avenue
Lewisburg, Pennsylvania 17837 USA

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ABSTRACT

A digroup is an algebra defined on a set having two associative binary operations, $\vdash$ and $\dashv$. Digroups play an important role in an open problem in the theory of Leibniz algebras. We present a brief overview of digroups and a set of more general axioms for a digroup than used previously. We then consider several properties of a digroup having distinct elements a and b such that $a \vdash b = b \vdash a$, but $a \vdash b \neq a \dashv b$.

I. INTRODUCTION

A vexing problem in the theory of Leibniz algebra is finding a generalization of Lie's third theorem. Lie's third theorem associates a (local) Lie group to any Lie algebra, real or complex. Seeking appropriate analogues of Lie groups for Leibiniz algebras, Loday [1] became so exasperated at one point that he dubbed these objects *coquecigrues*—absurdity incarnate. Kinyon [2] found a partial solution using digroups (defined below), as follows: if a Leibniz algebra splits over an ideal containing the ideal generated by squares, then it is isomorphic to the tangent algebra of a linear Lie digroup. In particular, if the split Leibniz algebra is actually a Lie algebra, then the linear Lie digroup coincides with the Lie group from Lie's third theorem. For Leibniz algebras that do not split, the problem remains open.

Loday was the first to use the idea of a digroup [3]; since then others have refined Loday's work, e.g., [4] and [5]. Kinyon modified Loday's definition of digroup to give one that is much cleaner (see below). As part of his partial solution to the *coquecigrue* problem, he showed that every digroup is a product of a group and a trivial digroup. J. D. Phillips [6] gave an even simpler basis of axioms for a digroup. In this paper, we generalize Kinyon's first axiom of associativity to give yet a more general axiom scheme for digroups. Then we look at some properties of pairs of elements that commute under one of the binary operations in a digroup.

Though we will not make use of this definition in this paper, a *Lie digroup* is a smooth manifold with a digroup structure such that the digroup operations are smooth mappings. A *linear Lie digroup* is a Lie digroup recovered from the product of a Lie group and a module in a natural way [2].

II. AXIOMIZATIONS OF DIGROUPS

Kinyon's definition of a digroup is as follows.

Definition 1 (Kinyon). A digroup is a set, G, equipped with two binary operations, $\vdash$ and $\dashv$, with a unary operation, †, and with nullary operation, 1, satisfying each of the following six axioms:

G1. $(G, \vdash)$ and $(G, \dashv)$ are both semigroups.
G2. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$.
G3. $x \dashv (y \vdash z) = x \dashv y \dashv z$.
G4. $(x \dashv y) \vdash z = x \vdash y \vdash z$.
G5. $1 \vdash x = x = x \dashv 1$.
G6. $x \vdash x^\dagger = 1 = x^\dagger \dashv x$.

Note that a digroup is essentially a left group and a right group together with some compatibility axioms. J. D. Phillips [6] showed that *G2* through *G4* can be simplified as follows:

Theorem 1 (Phillips). A set, G, is a digroup, if and only if it is equipped with two binary operators, $\vdash$ and $\dashv$, with a unary operation, †, and with nullary operation, 1, satisfying the following four axioms:

G1. $(G, \vdash)$ and $(G, \dashv)$ are both semigroups.
G2* $x \vdash (x \dashv z) = (x \vdash x) \dashv z$.
G5. $1 \vdash x = x = x \dashv 1$.
G6. $x \vdash x^\dagger = 1 = x^\dagger \dashv x$.

We will show that *G1*, the associativity of $\vdash$ and $\dashv$, may be weakened to a form of the *Semi-Moufang* identity.

Theorem 2. A set, G, is a digroup if and only if it is closed under two binary operations $\vdash$ and $\dashv$, has a unary operation, †, and with nullary operation, 1, satisfying the following axioms:

*G1** $\vdash$ and $\dashv$ are semi-Moufang, *i.e.*,
$x \vdash ((y \vdash z) \vdash x) = (x \vdash y) \vdash (z \vdash x)$ and
$x \dashv ((y \dashv z) \dashv x) = (x \dashv y) \dashv (z \dashv x)$.
G2. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$.
G5. $1 \vdash x = x = x \dashv 1$.
G6. $x \vdash x^\dagger = 1 = x^\dagger \dashv x$.

Proof. All digroups satisfy these axioms because they are a weaker set of axioms. Conversely, let G be an algebra satisfying these axioms. Using Phillips' Axioms of a digroup, we need only prove associativity of the two binary operations, (since *G2* implies *G2** by setting y =x). We will show the proof for the associativity of $\vdash$; the proof for $\dashv$ is similar.

We introduce six auxiliary identities to help in proving $(x \vdash y) \vdash z = x \vdash (y \vdash z)$:

A1. $(x \vdash y) \vdash 1 = x \vdash (y \vdash 1)$.
A2. $1 \dashv x = x \vdash 1$.
A3. $x \vdash ((x^\dagger \vdash y) \vdash x) = y \vdash x$
A4. $x^\dagger \vdash 1 = x^\dagger$.
A5. $x \vdash (x^\dagger \vdash (y \vdash 1)) = y \vdash 1$.
A6. $(x \vdash 1) \vdash y = x \vdash y$.

We first show that G must satisfy *A1-A6* then show that $\vdash$ is associative.

A1: $(x \vdash y) \vdash 1 \overset{G5}{=} 1 \vdash [(x \vdash y) \vdash 1] \overset{G1*}{=} (1 \vdash x) \vdash (y \vdash 1) \overset{G5}{=} x \vdash (y \vdash 1)$.

A2: $1 \dashv x \overset{G6}{=} (x \vdash\vdash x^\dagger) \dashv x = \overset{G2}{=} x \vdash (x^\dagger \dashv x) \overset{G6}{=} x \vdash 1$.

A3: $x \vdash [(x^\dagger \vdash y) \vdash x] \overset{G1*}{=} (x \vdash x^\dagger) \vdash (y \vdash x) \overset{G6}{=} 1 \vdash (y \vdash x) \overset{G5}{=} y \vdash x$.

A4: $x^\dagger \vdash 1 \overset{G5}{=} x^\dagger \vdash (1 \vdash 1) \overset{A1}{=} (x^\dagger \vdash 1) \vdash 1 \overset{A2}{=} (1 \dashv x^\dagger) \vdash 1 \overset{A2}{=} 1 \dashv (1 \dashv x^\dagger)$
$\overset{G6}{=} (x^\dagger \dashv x) \dashv (1 \dashv x^\dagger) \overset{G1*}{=} x^\dagger \dashv [(x \dashv 1) \dashv x^\dagger] \overset{G5}{=} x^\dagger \dashv (x \dashv x^\dagger) \overset{G5}{=} (x^\dagger \dashv 1) \dashv (x \dashv x^\dagger)$
$\overset{G1*}{=} x^\dagger \dashv [(1 \dashv x) \dashv x^\dagger] \overset{A2}{=} x^\dagger \dashv [(x \vdash 1) \dashv x^\dagger] \overset{G2}{=} x^\dagger \dashv [x \vdash (1 \dashv x^\dagger)] \overset{A2}{=} x^\dagger \dashv [x \vdash (x^\dagger \vdash 1)]$
$\overset{A1}{=} x^\dagger \dashv [(x \vdash x^\dagger) \vdash 1] \overset{G6}{=} x^\dagger \dashv (1 \vdash 1) \overset{G5}{=} x^\dagger$.

A5:

$$x \vdash [x^{\dagger} \vdash (y \vdash 1)] \overset{A2}{=} x \vdash [x^{\dagger} \vdash (1 \dashv y)] \overset{G2}{=} x \vdash [(x^{\dagger} \vdash 1) \dashv y] \overset{A2}{=} x \vdash [(1 \dashv x^{\dagger}) \dashv y]$$

$$\overset{G5}{=} x \vdash [(1 \dashv x^{\dagger}) \dashv (y \dashv 1)] \overset{G1*}{=} x \vdash [1 \dashv (x^{\dagger} \dashv y) \dashv 1] \overset{G5}{=} x \vdash [1 \dashv (x^{\dagger} \dashv y)]$$

$$\overset{A2}{=} x \vdash [(x^{\dagger} \dashv y) \vdash 1] \overset{A1}{=} [x \vdash (x^{\dagger} \dashv y)] \overset{G2}{=} [(x \vdash x^{\dagger}) \dashv y] \vdash 1 \overset{G5}{=} (1 \dashv y) \vdash 1$$

$$\overset{A2}{=} (y \vdash 1) \vdash 1 \overset{A1}{=} y \vdash (1 \vdash 1) \overset{G5}{=} y \vdash 1.$$

The following lengthy operations show that G satisfies *A6*.

A6:

$$(x \vdash 1) \vdash y \overset{A3}{=} y \vdash ((y^{\dagger} \vdash (x \vdash 1)) \vdash y) \overset{A1}{=} y \vdash (((y^{\dagger} \vdash x) \vdash 1) \vdash y)$$

$$\overset{G1*}{=} (y \vdash (y^{\dagger} \vdash x)) \vdash (1 \vdash y) \overset{G5}{=} (y \vdash (y^{\dagger} \vdash x)) \vdash y \overset{A3}{=} y \vdash ((y^{\dagger} \vdash (y \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A4}{=} y \vdash (((y^{\dagger} \vdash 1) \vdash (y \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A5}{=} y \vdash ((((y^{\dagger} \vdash x) \vdash ((y^{\dagger} \vdash x)^{\dagger} \vdash (y^{\dagger} \vdash 1))) \vdash ((y \vdash (y^{\dagger} \vdash 1))) \vdash y)$$

$$\overset{A4}{=} y \vdash ((((y^{\dagger} \vdash x) \vdash ((y^{\dagger} \vdash x)^{\dagger} \vdash y^{\dagger})) \vdash (y\ y^{\dagger}\ (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{G1*}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((((y^{\dagger} \vdash x)^{\dagger} \vdash y^{\dagger}) \vdash y) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{G5}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((((y^{\dagger} \vdash x)^{\dagger} \vdash (1 \vdash y^{\dagger})) \vdash y) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A4}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((((y^{\dagger} \vdash x)^{\dagger} \vdash (1 \vdash (y^{\dagger} \vdash 1))) \vdash y) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A2}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((((y^{\dagger} \vdash x)^{\dagger} \vdash (1 \vdash (1 \dashv y^{\dagger}))) \vdash y) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{G2}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((((y^{\dagger} \vdash x)^{\dagger} \vdash ((1 \vdash 1) \dashv y^{\dagger})) \vdash y) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{G2}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((((y^{\dagger} \vdash x)^{\dagger} \vdash (1 \vdash 1)) \dashv y^{\dagger}) \vdash y) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A1}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((((((y^{\dagger} \vdash x)^{\dagger} \vdash 1) \vdash 1) \dashv y^{\dagger}) \vdash y) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{G2}{=} y \vdash (((y^{\dagger} \vdash x) \vdash (((((y^{\dagger} \vdash x)^{\dagger} \vdash 1) \vdash (1 \dashv y^{\dagger})) \vdash y) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A2}{=} y \vdash (((y^{\dagger} \vdash x) \vdash (((((y^{\dagger} \vdash x)^{\dagger} \vdash 1) \vdash (y^{\dagger} \vdash 1)) \vdash y) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A4}{=} y \vdash (((y^{\dagger} \vdash x) \vdash (((((y^{\dagger} \vdash x)^{\dagger} \vdash 1) \vdash y^{\dagger}) \vdash y) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A3}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((y \vdash ((y^{\dagger} \vdash (((y^{\dagger} \vdash x)^{\dagger} \vdash 1) \vdash y^{\dagger})) \vdash y)) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A4}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((y \vdash ((y^{\dagger} \vdash (((y^{\dagger} \vdash x)^{\dagger} \vdash 1) \vdash (y^{\dagger} \vdash 1))) \vdash y) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A2}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((y \vdash ((y^{\dagger} \vdash (((y^{\dagger} \vdash x)^{\dagger} \vdash 1) \vdash (1 \dashv y^{\dagger}))) \vdash y) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{G2}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((y \vdash ((y^{\dagger} \vdash ((((y^{\dagger} \vdash x)^{\dagger} \vdash 1) \vdash 1) \dashv y^{\dagger})) \vdash y)) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{G2}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((y \vdash (((y^{\dagger} \vdash (((y^{\dagger} \vdash x)^{\dagger} \vdash 1) \vdash 1)) \dashv y^{\dagger}) \vdash y)) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A1}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((y \vdash ((((y^{\dagger} \vdash ((y^{\dagger} \vdash x)^{\dagger} \vdash 1)) \vdash 1) \dashv y^{\dagger}) \dashv y)) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{G2}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((y \vdash (((y^{\dagger} \vdash ((y^{\dagger} \vdash x)^{\dagger} \vdash 1)) \vdash (1 \dashv y^{\dagger})) \vdash y)) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A2}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((y \vdash (((y^{\dagger} \vdash ((y^{\dagger} \vdash x)^{\dagger} \vdash 1)) \vdash (y^{\dagger} \vdash 1)) \vdash y)) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A4}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((y \vdash (((y^{\dagger} \vdash ((y^{\dagger} \vdash x)^{\dagger} \vdash 1)) \vdash y^{\dagger}) \vdash y)) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{G1*}{=} y \vdash (((y^{\dagger} \vdash x) \vdash (((y \vdash (y^{\dagger} \vdash ((y^{\dagger} \vdash x)^{\dagger} \vdash 1))) \vdash (y^{\dagger} \vdash y)) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A5}{=} y \vdash (((y^{\dagger} \vdash x) \vdash ((((y^{\dagger} \vdash x)^{\dagger} \vdash (y^{\dagger} \vdash y)) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A4}{=} y \vdash (((y^{\dagger} \vdash x) \vdash (((y^{\dagger} \vdash x)^{\dagger} \vdash (y^{\dagger} \vdash y)) \vdash (y^{\dagger} \vdash x))) \vdash y)$$

$$\overset{A3}{=} y \vdash (((y^{\dagger} \vdash y) \vdash (y^{\dagger} \vdash x)) \vdash y) \overset{G1*}{=} (y \vdash (y^{\dagger} \vdash y)) \vdash ((y^{\dagger} \vdash y)) \vdash ((y^{\dagger} \vdash x) \vdash y)$$

$$\overset{A4}{=} (y \vdash ((y^{\dagger} \vdash 1) \vdash y)) \vdash ((y^{\dagger} \vdash x) \vdash y) \overset{A3}{=} (1 \vdash y) \vdash ((y^{\dagger} \vdash x) \vdash y)$$

$$\overset{G5}{=} y \vdash ((y^{\dagger} \vdash x) \vdash y) \overset{A3}{=} x \vdash y.$$

We now show that $\vdash$ is associative.

$$x \vdash (y \vdash z)$$

$$\overset{A6}{=} (x \vdash 1) \vdash (y \vdash z)$$

$$\overset{A6}{=} (x \vdash 1) \vdash ((y \vdash 1) \vdash z)$$

$$\overset{A5}{=} (z \vdash (z^{\dagger} \vdash (x \vdash 1)) \vdash ((y \vdash 1) \vdash z)$$

$$\overset{G1*}{=} z \vdash (((z^{\dagger} \vdash (x \vdash 1)) \vdash (y \vdash 1)) \vdash z)$$

$$\overset{A2}{=} z \vdash (((z^{\dagger} \vdash (x \vdash 1)) \vdash (1 \dashv y)) \vdash z)$$

$$\overset{G2}{=} z \vdash (((z^{\dagger} \vdash (x \vdash 1)) \dashv y) \vdash z)$$

$$\overset{A1}{=} z \vdash (((z^{\dagger} \vdash ((x \vdash 1) \vdash 1)) \dashv y) \vdash z)$$

$$\overset{G2}{=} z \vdash ((z^{\dagger} \; (((x \vdash 1) \vdash 1) \dashv y)) \vdash z)$$

$$\overset{G2}{=} z \vdash ((z^{\dagger} \vdash ((x \vdash 1) \vdash (1 \dashv y))) \vdash z)$$

$$\overset{A2}{=} z \vdash ((z^{\dagger} \vdash ((x \vdash 1) \vdash (y \vdash 1))) \vdash z)$$

$$\overset{A3}{=} ((x \vdash 1) \vdash (y \vdash 1)) \vdash z$$

$$\overset{A2}{=} ((x \vdash 1) \vdash (1 \dashv y)) \vdash z$$

$$\overset{G2}{=} (((x \vdash 1) \vdash 1) \dashv y) \vdash z$$

$$\overset{A1}{=} ((x \vdash (1 \vdash 1)) \dashv y) \vdash z$$

$$\overset{G2}{=} (x \vdash ((1 \vdash 1) \dashv y)) \vdash z$$

$$\overset{G2}{=} (x \vdash (1 \vdash (1 \dashv y))) \vdash z$$

$$\overset{A2}{=} (x \vdash (1 \vdash (y \vdash 1))) \vdash z$$

$$\overset{G5}{=} (x \vdash (y \vdash 1) \vdash z$$

$$\overset{A1}{=} ((x \vdash y) \vdash\vdash 1) \vdash z$$

$$\overset{A6}{=} (x \vdash y) \vdash z.$$

□

III. COMMUTING PAIRS UNDER $\vdash$

a. Example

Suppose G is a digroup that is not a group, and further suppose that G has distinct elements a and b such that $a \vdash b = b \vdash a$ and $a \vdash b \neq a \dashv b$. We will show that the order of G must be composite, and further, that the set of pairs that commute under $\vdash$ including those of the form (x, x) is a sub-right group of G×G.

The following is the smallest such digroup of odd order found by the computer program *Mace 4*.

Let $G = \mathbb{Z}_{15}$. We define $\vdash$ and $\dashv$ as follows:

⊢	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	11	2	3	1	7	10	0	8	4	5	9	6	13	14	12
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	11	2	3	1	7	10	0	8	4	5	9	6	13	14	12
3	6	3	1	2	8	9	11	4	7	10	5	0	14	12	13
4	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
5	6	3	1	2	8	9	11	4	7	10	5	0	14	12	13
6	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
7	11	2	3	1	7	10	0	8	4	5	9	6	13	14	12
8	6	3	1	2	8	9	11	4	7	10	5	0	14	12	13
9	11	2	3	1	7	10	0	8	4	5	9	6	13	14	12
10	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
11	6	3	1	2	8	9	11	4	7	10	5	0	14	12	13
12	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
13	11	2	3	1	7	10	0	8	4	5	9	6	13	14	12
14	6	3	1	2	8	9	11	4	7	10	5	0	14	12	13

⊣	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	5	0	5	4	0	4	0	5	4	5	0	4	0	5	4
1	2	1	2	3	1	3	1	2	3	2	1	3	1	2	3
2	3	2	3	1	2	1	2	3	1	3	2	1	2	3	1
3	1	3	1	2	3	2	3	1	2	1	3	2	3	1	2
4	0	4	0	5	4	5	4	0	5	0	4	5	4	0	5
5	4	5	4	0	5	0	5	4	0	4	5	0	5	4	0
6	9	6	9	8	6	8	6	9	8	9	6	8	6	9	8
7	11	7	11	10	7	10	7	11	10	11	7	10	7	11	10
8	6	8	6	9	8	9	8	6	9	6	8	9	8	6	9
9	8	9	8	6	9	6	9	8	6	8	9	6	9	8	6
10	7	10	7	11	10	11	10	7	11	7	10	11	10	7	11
11	10	11	10	7	11	7	11	10	7	10	11	7	11	10	7
12	13	12	13	14	12	14	12	13	14	13	12	14	12	13	14
13	14	13	14	12	13	12	13	14	12	14	13	12	13	14	12
14	12	14	12	13	14	13	14	12	13	12	14	13	14	12	13

Here $9 \vdash 5 = 5 \vdash 9 = 10 \neq 6 = 9 \dashv 5$.

b. The Order of G

We first collect some basic definitions and results about right groups that we will need to prove our results (Lemma 1). Note that left groups have results corresponding to those listed for right groups.

Lemma 1. [2] Let $(G, \vdash)$ be a right group, let $E = \{e \in G | e \vdash x = x\}$, and let $J = \{x^{-1} | x \in G\}$.

1. $(x \vdash y)^{-1} = y^{-1} \vdash x^{-1}$ for all $x, y \in G$.
2. $x \vdash 1 = (x^{-1})^{-1}$ for all $x \in G$.
3. E is a right zero semigroup.
4. J is a group.
5. $G = J \vdash E \cong (J \times E, \vdash)$.
6. G is a group if and only if $\vdash = \dashv$ if and only if $E = \{1\}$.

Theorem 3. If G is a digroup with $a, b \in G$ satisfying $a \vdash b = b \vdash a$ and $a \vdash b \neq a \dashv b$, the $|G|$ is composite.

Proof. Let $(G, \vdash, \dashv)$ be a digroup and suppose $|G| = p$ for some prime p. From Lemma 1 we know that $G \cong (J \times E, \vdash)$. Therefore $|E| = 1$ or $|E| = p$. Suppose $|E| = 1$. Then by Lemma 1, $\vdash = \dashv$ and therefore

$a \vdash b = a \dashv b$, a contradiction. Suppose $|E| = p$. Then $\forall_{g_i}, g_j \in G$, $g_i \vdash g_j = g_j$. Therefore, as illustrated by the Cayley table below, there are no commutative pairs in $(G, \vdash)$.

$\vdash$	g_1	g_2	g_3	$\cdots$	g_p
g_1	g_1	g_2	g_3	...	g_p
g_2	g_1	g_2	g_3	...	g_p
g_3	g_1	g_2	g_3	...	g_p
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
g_p	g_1	g_2	g_3	...	g_p

□

c. Commuting Pairs Under $\vdash$

Here we list all the pairs from Example 1 that commute under $\vdash$, excluding pairs of the form (x, x), and we also show $J \vdash E$. Here $J = \{1, 2, 3\} \cong \mathbb{Z}_3$ and $E = \{1, 4, 6, 10, 12\}$.

Commutative Pairs Under $\vdash$

(1,2)	(1,3)	(2,3)
(4,7)	(4,8)	(7,8)
(6,0)	(6,11)	(0,11)
(10,5)	(10,9)	(5,9)
(12,13)	(12,14)	(13,14)

$J \vdash E$

$3 \vdash 1 = 3$	$2 \vdash 1 = 2$	$1 \vdash 1 = 1$
$3 \vdash 4 = 8$	$2 \vdash 4 = 7$	$1 \vdash 4 = 4$
$3 \vdash 6 = 11$	$2 \vdash 6 = 0$	$1 \vdash 6 = 6$
$3 \vdash 10 = 5$	$2 \vdash 10 = 9$	$1 \vdash 10 = 10$
$3 \vdash 12 = 14$	$2 \vdash 12 = 13$	$1 \vdash 12 = 12$

From these tables we notice two things. First, going across a row of $J \vdash E$, all elements commute, *e.g.*, 8 and 7 commute, 4 and 7 commute, and 4 and 8 commute. Second, going down a column of $J \vdash E$, no elements commute, either with elements of their own column or with elements of other columns not also in the same row, *e.g.*, 11 and 7 do not commute. We can generalize these observations as follows.

For each of the following lemmas, let $j_x \in J$ and let $e_x \in E$.

Lemma 2. If J is abelian and *e* is a fixed element of E, then all pairs of the form $(j_k \vdash e, j_l \vdash e)$ commute.

Proof. Assume J is abelian and choose *e* a fixed element of E. Since $J \leq G$, $j_k \vdash j_l = j_{kl} \in J$. Therefore $(j_k \vdash e) \vdash (j_l \vdash e) = j_k \vdash (e \vdash j_l) \vdash e = j_{kl} \vdash e = j_{l\,k} \vdash e = (j_l \vdash e) \vdash (j_k \vdash e)$.

Lemma 3. Define ~ by a ~ b if $a = j_k \vdash e$ and $b = j_l \vdash e$ for the same e. Clearly ~ is an equivalence relation.

Lemma 4. No pairs of the form $(j_k \vdash e_1, j_l \vdash e_2)$ commute, where $e_1 \neq e_2$.

Proof. $(j_k \vdash e_1) \vdash (j_l \vdash e_2) = j_{kl} \vdash e_2$, while $(j_l \vdash e_2) \vdash (j_k \vdash e_1) = j_{l\,k} \vdash e_1$. From Lemma 3, we know that $j_{kl} \vdash e_2 \neq j_{l\,k} \vdash e_1$.

□

There is one further lemma that will be useful to us.

Lemma 5. $\exists$ a, b $\in (G, \vdash, \dashv)$ satisfying $a \vdash b = b \vdash a$ if and only if $a^{-1} \vdash b^{-1} = b^{-1} \vdash a^{-1}$.

Proof. Let G be a digroup.

$\Rightarrow$Let a, b $\in$ G such that $a \vdash b = b \vdash a$. Then $b^{-1} \vdash a^{-1} = (a \vdash b)^{-1} = (b \vdash a)^{-1} = a^{-1} \vdash b^{-1}$.

$\Leftarrow$Let a, b $\in$ G such that $a^{-1} \vdash b^{-1} = b^{-1} \vdash a^{-1}$. Then $(b \vdash a)^{-1} = a^{-1} \vdash b^{-1} = b^{-1} \vdash a^{-1} = (a \vdash b)^{-1}$.

□

Definition 2. Let $H = \{(a, b) \in (G \times G, \vdash) | \; a \vdash b = b \vdash a\}$. For all (a, b), (c, d) $\in$ H, define $(a,b) \vdash (c,d)$ by $(a,b) \vdash (c,d) = (a \vdash c, b \vdash d)$.

Theorem 4. If J is an abelian subgroup of a digroup G, then H is a sub-right group of $(G \times G, \vdash)$.

Proof. Let G be a digroup. Clearly $(G \times G, \vdash)$ is a right group. Let J be an abelian subgroup of G, and let $(a,b), (c,d) \in H$.

Closure: From Lemma 4 we know we can write (a,b) as $(j_a \vdash e_a, j_b \vdash e_a)$ and (c,d) as $(j_c \vdash e_c, j_d \vdash e_c)$. Therefore $(a,b) \vdash (c,d) = (j_a \vdash e_a \vdash j_c \vdash e_c, j_b \vdash e_a \vdash j_d \vdash e_c) = (j_{ac} \vdash e_c, j_{bd} \vdash e_c)$, which commutes since J is abelian. Therefore H is closed under $\vdash$.

Right Inverses: From Lemma 5, we know that $(a^{-1}, b^{-1}) \in H$. Obviously $(a,b) \vdash (a^{-1},b^{-1}) = (1,1) \in H$.

Left Identity: Clearly $(1,1) \vdash (a,b) = (a,b)$.

□

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