

A Different Description of a Family of Middle- α Cantor Sets

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ABSTRACT

In this note we give an explicit description of a family of middle- α Cantor sets, with $\alpha = (q - 2)/q$ and $q = 3, 4, 5, \dots$

I. INTRODUCTION

Georg Cantor (1845-1918), the founder of axiomatic set theory studied many interesting sets. He was very interested in infinite sets, in particular those with strange properties. One of the sets that he constructed is named after him: the middle-third Cantor set. The middle third Cantor set C is the set of all $x \in [0, 1]$ with the ternary expansion

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n},$$

with $a_n = 0, 2$ for all $n \in \mathbb{N}$.

This set and its construction is an interesting and amazing set for any sophomore student in analysis. It is uncountable, perfect, compact, and has some other nice properties which makes it a curious set. On the other hand, the construction of the set shows that the role of the middle third of the interval can be replaced by any $\alpha \in (0, 1)$, and a straightforward generalization of this set, the middle- α Cantor set can be obtained [1]. To define a middle- α Cantor set in the interval $[0, 1]$, let $\alpha \in (0, 1)$ and $\beta = (1 - \alpha)/2$, then the middle- α Cantor set Γ_α is the set of all $x \in [0, 1]$ with expansion

$$x = \sum_{n=0}^{\infty} a_n \beta^n (1 - \beta),$$

And $a_n = 0, 1$ for all $n \in \mathbb{N}_0$. We notice that when $\alpha = \beta = 1/3$ this representation is equivalent to the representation of points in the middle third Cantor set. By using the idea of affine maps we can have an equivalent definition of the Γ_α [1]. For this, with the same assumptions on α and β , define the maps

$$T_0(x) = \beta x$$

and

$$T_1(x) = \beta x + (1 - \beta)$$

Now let $I_0 = [0, 1]$ and for $n \geq 1$, define I_n inductively as follows:

$$I_n = T_0(I_{n-1}) \cup T_1(I_{n-1})$$

Since T_0 and T_1 are continuous and closed, they take each closed subinterval of $[0, 1]$ into a closed subinterval of $[0, 1]$, and each I_n is a disjoint union of 2^n closed subintervals of $[0, 1]$ which are the image of 2^{n-1} closed subintervals of $[0, 1]$ under T_0, T_1 . To illustrate how these intervals are constructed, let's look at them more closely. By the definition of T_0 and T_1 , $T_0(I_0) = [0, \beta]$ and $T_1(I_0) = [1 - \beta, 1]$, respectively, and by the definition of I_1 ,

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$$I_1 = T_0(I_0) \cup T_1(I_0) = [0, \beta] \cup [1 - \beta, 1]$$

$$T_1(I_1) = [1 - \beta, \beta^2 + (1 - \beta)] \cup [(1 - \beta) + \beta(1 - \beta), \beta],$$

The removed interval has length $1 - 2\beta = \alpha$.
In the same way,

$$T_0(I_1) = [0, \beta^2] \cup [\beta(1 - \beta), \beta]$$

and

$$I_2 = T_0(I_1) \cup T_1(I_1) = [0, \beta^2] \cup [\beta - \beta^2, \beta] \cup [1 - \beta, 1 - \beta + \beta^2] \cup [1 - \beta^2, 1]$$

In general, each map T_0, T_1 sends 2^{n-1} disjoint closed intervals into 2^{n-1} disjoint closed intervals, hence their union, which is I_n , is the disjoint union of 2^n disjoint closed intervals of length β^n . We get I_n by removing from the middle of each component of I_{n-1} an open interval of length $\alpha\beta^{n-1}$. By the construction of I_n ,

$$I_0 \supset I_1 \supset I_2 \supset I_3 \supset \dots$$

and each I_n is a closed, and hence compact, subset of $[0, 1]$. This collection has the finite intersection property so again by the compactness of $[0, 1]$ they have a nonempty intersection. Then the middle- α Cantor set is defined as:

$$\Gamma_\alpha = \bigcap_{n=0}^{\infty} I_n.$$

II. THE MAIN RESULT

The following theorem gives an explicit description of a family of Γ_α s.

each of four components has length β^2 , and the length of the removed interval from each component of I_1 is $\beta(1 - 2\beta) = \alpha\beta$, hence

Theorem: Let Γ_α be a middle- α Cantor set. Then

$$\Gamma_\alpha = [0, 1] - \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{q^m-1} \left(\frac{qk+1}{q^m}, \frac{qk+(q-1)}{q^m} \right)$$

for $\alpha = \frac{q-2}{q}$ and $q = 3, 4, 5, \dots$

To prove this theorem, we need the following two lemmas. Before stating them and their proofs let

$$I_n^* = I_0 - I_n \text{ for } n = 1, 2, \dots$$

a. Lemma 1

With the above definitions and notations, the equality

$$I_n^* = T_0(I_{n-1}^*) \cup I_1^* \cup T_1(I_{n-1}^*)$$

holds for every $n \geq 1$.

Proof: By the definition of I_n^* :

$$\begin{aligned} I_n^* &= I_0 - I_n \\ &= I_0 - (T_0(I_{n-1}) \cup T_1(I_{n-1})) \\ &= (I_0 - T_0(I_{n-1})) \cap (I_0 - T_1(I_{n-1})) \\ &= (I_0 - T_0(I_0 - I_{n-1}^*)) \cap (I_0 - T_1(I_0 - I_{n-1}^*)) \\ &= (I_0 - (T_0(I_0) - T_0(I_{n-1}^*))) \cap (I_0 - (T_1(I_0) - T_1(I_{n-1}^*))) \\ &= ((I_0 - T_0(I_0)) \cup T_0(I_{n-1}^*)) \cap ((I_0 - T_1(I_0)) \cup T_1(I_{n-1}^*)) \\ &= ((I_0 - T_0(I_0)) \cap (I_0 - T_1(I_0))) \cup ((I_0 - T_0(I_0)) \cap T_1(I_{n-1}^*)) \cup \\ &\quad (T_0(I_{n-1}^*) \cap (I_0 - T_1(I_0))) \cup (T_0(I_{n-1}^*) \cap T_1(I_{n-1}^*)) \\ &= T_1(I_{n-1}^*) \cup I_1^* \cup T_0(I_{n-1}^*) \end{aligned}$$

b. Lemma 2

With the above definitions and notations, the equality

$$I_n^* = \bigcup_{m=1}^n \bigcup_{k=0}^{q^m-1} \left(\frac{qk+1}{q^m}, \frac{qk+(q-1)}{q^m} \right)$$

holds for every $n \geq 1$.

$$\begin{aligned} I_n^* &= T_0(I_{n-1}^*) \cup I_1^* \cup T_1(I_{n-1}^*) \\ &= \left(\bigcup_{m=1}^{n-1} \bigcup_{k=0}^{q^m-1} \left(\frac{qk+1}{q^{m+1}}, \frac{qk+q+1}{q^{m+1}} \right) \right) \cup \left(\frac{1}{q}, \frac{q-1}{q} \right) \cup \left(\bigcup_{m=1}^{n-1} \bigcup_{k=q^m-q^{m-1}}^{q^m-1} \left(\frac{qk+1}{q^{m+1}}, \frac{qk+q-1}{q^{m+1}} \right) \right) \\ &= \left(\bigcup_{m=1}^{n-1} \bigcup_{k=0}^{q^m-1} \left(\frac{qk+1}{q^{m+1}}, \frac{qk+q-1}{q^{m+1}} \right) \right) \cup \left(\frac{1}{q}, \frac{q-1}{q} \right) \cup \left(\bigcup_{m=1}^{n-1} \bigcup_{k=q^m-q^{m-1}}^{q^m-1} \left(\frac{qk+1}{q^{m+1}}, \frac{qk+q-1}{q^{m+1}} \right) \right) \\ &\quad \cup \left(\bigcup_{m=1}^{n-1} \bigcup_{k=q^m-q^{m-1}}^{q^m-1} \left(\frac{qk+1}{q^{m+1}}, \frac{qk+q-1}{q^{m+1}} \right) \right) \\ &= \left(\frac{1}{q}, \frac{q-1}{q} \right) \cup \left(\bigcup_{m=1}^{n-1} \bigcup_{k=0}^{q^m-1} \left(\frac{qk+1}{q^{m+1}}, \frac{qk+q-1}{q^{m+1}} \right) \right) \\ &= \bigcup_{m=1}^n \bigcup_{k=0}^{q^m-1} \left(\frac{qk+1}{q^m}, \frac{qk+q-1}{q^m} \right). \end{aligned}$$

Now by virtue of Lemma 2, we prove the theorem:

$$\begin{aligned} \Gamma_\alpha &= \bigcap_{n=0}^\infty I_n \\ &= \bigcap_{n=0}^\infty (I_0 - I_n^*) \\ &= I_0 - \bigcup_{n=0}^\infty I_n^* \\ &= I_0 - \bigcup_{n=1}^\infty I_n^* \\ &= I_0 - \bigcup_{m=1}^\infty \bigcup_{k=0}^{q^m-1} \left(\frac{qk+1}{q^m}, \frac{qk+q-1}{q^m} \right) \end{aligned}$$

for every

$$\alpha = \frac{q-2}{q}, \quad q = 3, 4, 5, \dots$$

c. Corollary

The above theorem gives the ordinary Cantor set.

Proof. It is sufficient to put $q=3$:

$$\Gamma_{\frac{1}{3}} = [0, 1] - \bigcup_{m=1}^\infty \bigcup_{k=0}^{3^m-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right)$$

d. Question

As we mentioned in the introduction, the middle- α Cantor set can be defined for any $\alpha \in (0, 1)$ and even more for any closed interval $[a, b]$, $a < b$. In this note, we

Proof. We prove the assertion by induction. It is clear that it holds for $n = 1$. Let it hold for positive integer $n-1$, then Lemma 1 and the inclusion

$$\bigcup_{m=1}^{n-1} \bigcup_{k=q^{m-1}}^{q^m-q^{m-1}-1} \left(\frac{qk+1}{q^{m+1}}, \frac{qk+q-1}{q^{m+1}} \right) \subseteq \left(\frac{1}{q}, \frac{q-1}{q} \right)$$

imply that:

obtained an explicit formula for these sets, for

$$\alpha = \frac{q-2}{q}$$

and $q = 3, 4, 5, \dots$ Now a natural question to raise is: Can we construct these sets whenever

$$\alpha = \frac{q-2p}{q}$$

with $(p, q) = 1$, $q - 2p > 0$ and $p, q \in \{3, 4, 5, \dots\}$?

REFERENCE

1. R. Kraft. "What's the difference between Cantor sets?" *Amer. Math. Monthly* 101(1994), no.7, 640-650.



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