

Noetherian Rings—Dimension and Chain Conditions

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ABSTRACT

In this paper we look at the properties of modules and prime ideals in finite dimensional noetherian rings. This paper is divided into four sections. The first section deals with noetherian one-dimensional rings. Section Two deals with what we define a “zero minimum rings” and explores necessary and sufficient conditions for the property to hold. In Section Three, we come to the minimal prime ideals of a noetherian ring. In particular, we express noetherian rings with certain properties as finite direct products of noetherian rings with a unique minimal prime ideal, as an analogue to the expression of an artinian ring as a finite direct product of artinian local rings. Besides, we also consider the set of ideals I in R such that $M \neq IM$ for a given module M and show that a maximal element among these is prime. In Section Four, we deal with dimensions of prime ideals, Krull’s Small Dimension Theorem and generalize it (and its converse) to the case of a finite set of prime ideals. Towards the end of the paper, we also consider the sets of linear dependencies that might hold between the generators of an ideal and consider the ideals generated by the coefficients in such linear relations.

I. ALL RINGS ARE ASSUMED TO COMMUTATIVE RINGS WITH IDENTITY

Our main purpose is to study the noetherian rings of finite dimension and their modules. In this respect we shall prove results on the decomposition of a noetherian ring as a direct product of simpler noetherian rings and also consider if and only if conditions on the finitely generated modules of the ring that determine the dimension of the ring. We will also take up separately the question of dimension of a prime ideal in relation to the number of its generators and also establish a converse to these results [1-4].

Although artinian rings are defined apparently in a dual fashion (descending chain condition) to noetherian rings (ascending chain condition), it turns out that the artinian rings are just the noetherian zero dimensional rings (see [5]). Therefore,

we shall skip the case of dimension zero and start Section 1 with noetherian one-dimensional rings.

II. SECTION 1

We start with 1-dimensional noetherian rings. Suppose that M is a finitely generated R -module. We will consider the support of M , i.e., the maximal ideals m such that $M_m \neq 0$. Our first proposition will deal with the case of noetherian one-dimensional rings. Then we have the following result.

Proposition: Let R be a domain. The following statements are equivalent:

- 1) R is noetherian and 1-dimensional.
- 2) If M is a finitely generated R -module, M is either of finite length or $\text{Supp}(M) = \text{Spec}(R)$. (The Support $\text{Supp}(M)$ of a module is the set of all

prime ideals p of R such that $M_p \neq 0$.
The Spec of a ring refers to a set of all prime ideals of the ring.)

Proof: Let R be noetherian and 1-dimensional. Let M be a finitely generated R -module. We take a filtration of M such that the quotients of consecutive submodules in it are isomorphic to some R/p . If all of them are of the form R/m for some maximal ideal m , M comes out to be of finite length. If any of them is not maximal, it must be zero (since the ring is 1-dimensional). As such, one of the quotients is isomorphic to R . This quotient cannot become 0 under any localization whatsoever and hence $\text{Supp}(M) = \text{Spec}(R)$.

Conversely, let us assume (2). Take a non-zero ideal I in R . Then R/I is finitely generated. However, if we localize with respect to the prime ideal $0(R)$ a domain, we have generated $(R/I)_0 = 0$. Thus, $\text{Supp}(R/I)$ is not the same as $\text{Spec}(R)$. Thus R/I is of finite length and hence noetherian. Now suppose that there is an infinite increasing chain of ideals in R : say $I_0 \subset I_1 \subset I_2 \subset \dots$. Then if we take a non-zero ideal from this chain, say I_i , we see that R/I_i is non-noetherian. This is a contradiction. Hence, R is noetherian. Now, we take a non-zero prime ideal p in R . Consider R/p which is of finite length as an R/p module. Thus R/p is an artinian ring. Hence 0 , which is a prime ideal in R/p , is maximal. Thus, p is a maximal ideal. Hence R is 1-dimensional.

We now consider another equivalent condition for 1-dimensional noetherian domains. Accordingly, we have the following result.

Proposition: Let R be a domain. Then the following conditions are equivalent:

- 1) R is noetherian and one-dimensional.
- 2) Every finitely generated R -module of non-zero annihilator is of finite length.

Proof: Let M be an R -module of non-zero annihilator I . Then M may be considered as an R/I module. Since R/I is a 0-dimensional noetherian ring, M is of finite length.

Conversely, we consider the case $M = R/J$, where J is a non-zero ideal, as an R -module. We see that it has non-zero annihilator, i.e. J , and hence must be of finite length. Thus R/J is artinian, i.e.

noetherian and 0-dimensional. Hence, R is noetherian. Also, since R is a domain, (0) is a prime ideal. Thus R is 1-dimensional.

III. SECTION 2: ZERO MINIMUM RINGS

Let R be a noetherian ring. Then there might exist infinite descending chains of ideals in R . As an intermediate between noetherian and artinian rings, we will allow the descending chain condition to creep into the noetherian ring in a restricted manner. Consider all the ideals which may be written as the intersection of such an *infinite* descending chain. If R is non-artinian, this is a non-empty set. Let J be a maximal element of this set. Then R/J has a curious property. Any *infinite* descending chain of ideals in this ring has zero intersection. We now define:

Definition: *Zero minimum Ring*. A ring R is said to be a zero minimum ring if the intersection of an infinite decreasing chain of ideals in R is zero. (Note that R is not assumed to be noetherian.)

The following proposition is an analogue of the standard theorem which says that R is artinian. R is noetherian and 0 dimensional. However, here we have the condition that R is a domain. To see similar blends of chain conditions, the reader may consult [6].

Proposition: Let R be a domain. Then the following are equivalent:

- 1) R is noetherian and 1-dimensional.
- 2) R is a Zero-minimum Ring.

Proof: Suppose that R is noetherian and 1-dimensional. Take an infinite descending chain of ideals in R and consider their intersection, say J . If J is not 0, R/J is a noetherian ring having all its prime ideals maximal (the minimal prime ideal 0 has been eliminated by taking R/J). Hence R/J is an artinian ring and we have a decreasing infinite chain in R/J . This is clearly a contradiction.

Conversely, we now assume that R is a zero minimum ring. Then each R/J is artinian, for non-zero J . Thus, each R/J is noetherian for non-zero J , and by the same argument as in the previous result, this gives us that R is noetherian. Also, this tells us

that all non-zero prime ideals in R are maximal. Since R is a domain, we have one more prime ideal, namely 0, which makes the ring 1-dimensional.

The idea of studying noetherian 1-dimensional rings by studying their artinian quotient rings may also be traced back to [7]. In this respect, the reader may also consider the more recent [8]. The previous proof also says that if R is not a domain and at the same time is a zero minimum ring, we would have had: R is a noetherian ring and all non-zero prime ideals are maximal. Since 0 is not a prime ideal in R , we have:

Proposition: If R is not a domain, R is a zero minimum ring implies that it is artinian (and thus there is no infinite decreasing chain at all).

Proof: We see that R is a noetherian ring with all prime ideals maximal. Hence, R is artinian.

We will use this to prove the next result,

Proposition: Let R be a noetherian ring (and non-artinian) and let J be maximal among the ideals that can be written as the intersection of infinitely long, decreasing chains of ideals in R . Then J is prime.

Proof: Consider R/J . Note that R/J is noetherian but not artinian (there exists an infinite chain intersecting to 0). But R/J is a zero intersection ring. From the previous result, if R/J is not a domain, it must be artinian. Hence R/J is a domain. Thus, J is a prime ideal.

We end this section with the following theorem.

Proposition: If R is a zero minimum ring with non-zero Jacobson radical (the Jacobson radical is the intersection of all the maximal ideals of R), R must have *finitely many* maximal ideals.

Proof: Suppose that R has infinitely many maximal ideals. Choose countably many of these maximal ideals, say m_1, m_2, \dots . Consider the following chain of ideals:

$$m_1 \supset m_1 \cap m_2 \supset m_1 \cap m_2 \cap m_3 \dots$$

The inclusions are strict because: if there is equality at the k^{th} stage, $m_k \supset m_1 \cap m_2 \cap \dots \cap m_{k-1}$. This will imply that m_k contains some other maximal ideal, which is absurd. Thus,

there is an infinite chain. Since R is a zero-minimum ring, the intersection of this chain is (0). But, the intersection contains the Jacobson radical J . Thus, $J = 0$.

IV. SECTION 3: MINIMAL PRIME IDEALS

We shall now try to see the properties of the minimal prime ideals of a noetherian ring. It is well known that a *noetherian ring has finitely many prime ideals*, (see [5]). We shall use this property heavily in subsequent discussion. Further we note the following: if R is a noetherian ring and I an ideal in R , the ring R/I is also noetherian. Thus, it will also have finitely many prime ideals, i.e. there are only finitely many prime ideals minimal over a given prime ideal I . We shall take the liberty of referring to the prime ideals minimal over a given ideal I as the minimal primes of I rather than the minimal primes of R/I .

A remarkable fact about the artin ring is that it can be written as a finite direct product of artin local rings (see [5]). This is proved as follows (and the proof proceeds along the lines of the Chinese Remainder Theorem): The artin ring R has only finitely many maximal ideals, say m_1, \dots, m_n and the Jacobson radical J is nilpotent, say $J^k = 0$. Consider the product $\prod m_i^k$. Since the m_i are mutually co-prime, this product equals $\cap m_i^k$. But $\prod m_i^k \subseteq (\cap m_i)^k = (0)$. Hence we have the natural isomorphism from $R = R/\cap m_i^k$ to the direct product of the artinian local rings R/m_i^k . We note that in a 0-dimensional ring we might actually think of the maximal ideal as a minimal ideal. We would like to see some version of this property for higher dimensional rings as well. However, for this we see in the following proposition that we must make the assumption that the minimal prime ideals of the ring are actually co-prime. (The property of being mutually co-prime is obvious for the maximal ideals, but it has to be assumed when we are dealing with minimal prime ideals.)

Proposition: Let R be a finite dimensional noetherian ring such that the minimal prime ideals in R are actually co-prime. Then, if R is k -dimensional, R can be written as a finite direct product of noetherian k -dimensional rings, each having only one minimal prime ideal.

Proof: In the above proof we replace the Jacobson radical by the nilradical, which is obviously nilpotent. A ring of the form R/p^n , where p is a minimal prime ideal, has only one minimal prime ideal (if q is a prime ideal containing p^n , $p = \text{rad}(p^n) \subseteq \text{rad}(q) = q$).

We know from Nakayama's Lemma that if M is a finitely generated module and I is an ideal contained in the Jacobson radical J and $M = JM$, we have $M = 0$. We now consider all the ideals I in R which have the property that $M \neq IM$. If we assume that $M \neq 0$, we know that J is in this set. Let us consider any maximal element among those ideals; we will show that it is a prime ideal.

Proposition: Let R be a noetherian ring and M be a finitely generated non-zero R -module. Consider the set of all ideals I in R such that $M \neq IM$. This set is obviously non-empty since $M \neq 0$. Consider an ideal J maximal with respect to this property. Then J is prime.

Proof: Consider the module $N = M/JM$. This may be treated as an R/J module. Now suppose that J is not prime. Consider any prime ideal p containing J ; then p contains J strictly. But this means that $M = pM$. However, $M = pM$ implies that $M_p = 0$, applying Nakayama's lemma to M_p over R_p . Thus $(M/JM)_p = 0$ for each prime ideal in R/p . But then $M/JM = 0$ or $M = JM$. This is a contradiction.

V. SECTION 4: Krull's Theorem (Small Dimension Theorem) and Generalizations

We now have come to the legendary Small Dimension Theorem of Krull. It gives us an upper bound for the height of a prime ideal in a noetherian ring based on the number of its generators. We state the theorem as follows:

Small dimension Theorem. Let I be an ideal in a noetherian ring R and let p be a prime ideal in R minimal over I . Then if $\mu(I)$ denotes the number of generators for I , $\mu(I) \geq \text{ht}(p)$.

We refer the reader to [6] for a proof of the above theorem. We note that the above theorem tells us that the height of any prime ideal p in a noetherian ring is finite

(we apply the theorem with $I = p$). Further, this means that if the ring has only finitely many maximal ideals, then the supremum of the heights of the maximal ideals is also finite. This supremum is referred to as the dimension (or Krull dimension) of the ring. The theorem therefore shows that the Krull dimension of a semi-local noetherian ring is finite.

We consider next the extension of the small dimension theorem (and its converse) to finite sets of prime ideals. Let p_1, \dots, p_n be a set of prime ideals *with no order relations among them*. This means that none of the prime ideals p_i is contained in any of the primes p_j , $i \neq j$. Let their heights be r_1, \dots, r_n respectively; and let $r = \max(r_1, \dots, r_n)$. If I is an ideal generated by k elements such that each p_i is minimal over I , the $k \geq r$. This is obvious. Now we consider the converse. The converse of the small dimension theorem is as follows: *Given a prime ideal of height r , there exists an ideal I , generated by r elements, such that p is minimal over I .* Again we refer the reader to [6] for a proof of the theorem. Accordingly, we replace p by a finite set of prime ideals *with no order relations among them*. Now we claim that there should exist an ideal I generated by r elements such that each p_i is minimal over it. Here r is the maximum of the heights of the prime ideals p_i .

Proposition: Let p_1, \dots, p_n be a set of prime ideals in a noetherian ring *with no order relations among them*, i.e. none of the prime ideals p_i is contained in p_j for any $i \neq j$. Let $\text{ht}(p_i) = r_i$ and let $r = \max(r_1, \dots, r_n)$. Then there exists an ideal I generated by r elements such that each prime of the prime ideals p_i is minimal over I , i.e. the image of p_i is a minimal prime ideal of R/I .

Proof: We shall apply induction on r . If $r = 0$, each of the prime ideals p_i is a minimal prime ideal and hence we can take $I = (0)$, which is generated by 0 elements. Now, we assume the result to be true for $r - 1$.

Let $q_1, \dots, q_s, q_{s+1}, \dots, q_t$ be the minimal prime ideals of R (R being noetherian, this set is finite). Some of the p_i 's may be minimal prime ideals of R , let us assume that q_1, \dots, q_s are the minimal prime ideals among them. It is possible that $s = 0$. Take any p_i . Then p_i cannot be a subset of $\bigcup_{j=s+1}^t q_j$ (by prime avoidance). Take $a_i \in$

$p_i - \cup_{i=s+1}^t q_i$. Let $a = a_1 a_2 \dots a_n$. Consider the ring $R/(a)$. Each of the prime ideals p_i contains a . But, neither of the prime ideals q_{s+1}, \dots, q_t contains a since they do not contain any of a_1, \dots, a_n .

Consider those p_i for which $ht(p_i) = r$. Each p_i contains a minimal prime ideal and these minimal prime ideals cannot occur among the q_1, \dots, q_s because the p_i 's have no order relations among them. The minimal prime ideals contained among the p_i 's of height r all come from the ideals q_{s+1}, \dots, q_t . Since neither of the prime ideals q_{s+1}, \dots, q_t contain a , in the ring $R/(a)$, these prime ideals are excluded. Hence, the height of those primes p_i such that $ht(p_i)$ (in R) gets reduced by at least one. Hence, the height of each p_i in $R/(a)$ is $\leq r - 1$. Assume that $\max(ht(p_1), \dots, ht(p_n))$ is $k < r$ (in the ring $R/(a)$). By induction, we will have an ideal I in $R/(a)$, generated by k elements such that each p_i is minimal over it. Thus, in the ring R , each p_i will be minimal over the ideal generated by I and a . This ideal has $k+1$ generators. Since $ht(p_i) = r$ for at least one p_i , $r \leq k+1$ and we know that $k < r$. Thus $k = r+1$ and the ideal we have is generated by exactly r elements.

Take a prime ideal of height r in a noetherian ring. Then there exists an ideal I generated by r elements such that p is minimal over I . Let $I = (a_1, \dots, a_r)$. We want to know more about the generators a_i . More precisely, we want to know what relations they satisfy. If I is not free on a_1, \dots, a_r , there exist (non trivial) linear combinations of the a_i 's which are 0. Let I_1 be the ideal generated by the coefficients of a_1 in all such combinations. More precisely, consider all linear combinations $\sum x_i a_i$ of the a_i 's that are 0. Consider all the x_1 's appearing in these, that is the coefficients of a_1 . Let us designate the ideal generated by these x_1 's as I_1 . Similarly, consider the x_2 's appearing in these combinations, that is, the coefficients of a_2 . Let us designate the ideal generated by these x_2 's as I_2 . Similarly, we define I_3, \dots, I_r . We note that:

Proposition: The sum of ideals $I_1 + I_2 + \dots + I_r$ is contained inside p .

Proof: We note that the height of p in R is the same as its height in R_p . Also, p^e will remain a minimal prime ideal over I^e in R_p . Thus I^e should have at least r generators. If

I_1 is not contained in p , I_1^e contains 1 and hence, in R_p , we can write $a_1/1$ as a linear combination of $a_2/1, \dots, a_r/1$. Hence $I_1 \subseteq p$ and so also for I_2, \dots, I_r . Thus $I_1 + I_2 + \dots + I_r$ is contained in p .

Corollary: If a_i is a generator for the ideal I , $\text{Ann}(a_i) \subseteq p$.

Having considered the prime ideals minimal over a given ideal I , we come to the problem of determining the number of prime ideals minimal over a given ideal I . This is the same as determining the number of prime ideals minimal over a given radical ideal (for p contains I iff p contains the radical of I).

Proposition: Consider the products of prime ideals contained inside I , i.e. products of the form $p_1 p_2 \dots p_t$ contained in I , where p_i need not be all distinct, *although each p_i contains I* . Consider such a product having minimal number of distinct terms, say k terms. Then there are at most k many prime ideals minimal over I .

Proof: Suppose that $p_1 p_2 \dots p_n$ is the chosen product and it has k distinct terms. Consider a prime ideal p containing I . If each of the p_i 's has an element not contained in p , say x_i , we see that the product $x_1 x_2 \dots x_n$, though contained in I , is not contained in p . Thus $p \supseteq p_i$ for some i . If p is chosen to be minimal over I , we see that it must equal p_i . Thus I has at most k prime ideals minimal over it.

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