Exactness, Tor and Flat Modules Over a Commutative Ring

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ABSTRACT

In this paper, we principally explore flat modules over a commutative ring with identity. We do this in relation to projective and injective modules with the help of derived functors like Tor and Ext. We also consider an extension of the property of flatness and induce analogies with the "special cases" occurring in flat modules. We obtain some results on flatness in the context of a noetherian ring. We also characterize flat modules generated by one element and obtain a necessary condition for flatness of finitely generated modules.

I. DEFINITIONS

Let R be a commutative ring with identity and consider the modules over R.

- Exact sequence: A sequence of maps A→B→C is said to be exact at B if the image of the map "entering" B is equal to the kernel of the map "leaving" B.
- Functor: It is a map from the set of R-modules to itself.
- Exact functor: A functor T which when applied to all the terms of an exact sequence induces another exact sequence is said to be exact. The functor T in question should be additive, i.e. if f is a map from M to N, there should exist an induced map T(f) from T(M) to T(N).
- Projective modules: A module P having the following property: If p is a map from M *onto* N and f is a map from P to N, there exists a map g from P to M such that $p \circ g = f$. In other words, P is such that the functor Hom(P,_) is exact.
- Injective modules: A module Q having the following property: If i is a *one to one* map from K to M and f is a map from K to Q, there exists a map g from M to Q such that $g \circ I = f$. In other words, the functor Hom(_,Q) is exact.
- Flat modules: Those modules F for which the functor $\_\otimes F$ is exact are termed flat modules.
- Derived functors: Let T be an additive functor and N be an R-module. If the C_i's are all projective modules and the following sequence is exact: $\ldots. \to \ldots \to C_1 \to C_0 \to N \to 0$, then we have a projective resolution of N. The i^{th}-derived functor with respect to T is the homology module at $T(C_i)$, *i.e.*, the quotient of the kernel of the map leaving $T(C_i)$ to the image of the map entering it from $T(C_{i-1})$. It can be proved that the derived functor is determined uniquely up to isomorphism by any projective resolution [1-7].

II. FLATNESS

Let R be a commutative ring with identity [2]. Flatness may be defined through either of the following equivalent conditions:

(i) If P→Q is a monomorphism of R-modules, the induced map from $M\otimes P \to M\otimes Q$ is also a monomorphism.
(ii) The functor $\_\otimes M$ is exact.

We note the following interesting isomorphism:

$$\mathrm{Hom}(E,\mathrm{Hom}(F,G)) \cong \mathrm{Hom}(E\otimes F,G).$$[1]

From these we obtain the following results:

> *Result (a)*: Suppose that G is injective and we have the exact sequence: $0\to\mathrm{Hom}(M,G)\to\mathrm{Hom}(N,G)\to\mathrm{Hom}(P,G)\to 0$. If E is flat, the functor Hom(E,_) preserves the exactness of this sequence. Thus, the functor Hom(_,G) for injective G enables a flat module to do what a projective module does.
> This follows if we apply the functors E⊗_ and Hom(_,G) one after the other to the sequence $0\to P\to N\to M\to 0$. Each of these functors preserves the exactness of the sequence.
>
> *Result (b)*: The condition that Hom(E⊗_,G) is exact can be replaced by the condition that Hom(E,G) is injective. We will now prove that if Hom(E,G) is injective for all injective modules G, then E is flat.
>
> *Result (c)*: If E and F are projective, so is E⊗F. This is because the successive use of the two functors Hom(F,_) and Hom(E,_) can be replaced by the functor Hom(E⊗F,_).

Result (a) can also be used to characterize flat modules. If we assume that the functor Hom(E⊗_,G) is exact for all injective modules G, we can prove that E is flat.

Let M be an arbitrary R-module. We represent M as the quotient of a free module F. Thus we have the exact sequence $0\to K\to F\to M\to 0$. Tensoring with E, we have the exact sequence $0\to\mathrm{Tor}(M,E)\to E\otimes K\to E\otimes F\to E\otimes M\to 0$.

Let G be any injective module. Then $0 \to \mathrm{Hom}(E\otimes M,G) \to \mathrm{Hom}(E\otimes F,G) \to \mathrm{Hom}(E\otimes K,G) \to \mathrm{Hom}(\mathrm{Tor}(M,E),G) \to 0$ is exact. But, if we assume that Hom(E⊗_,G) is exact, we have Hom(Tor(M,E),G)=0 for all injective G. Now, Tor(M,E) can be embedded in some injective module G and hence we have Tor(M,E)=0 for all modules M. Thus, E is flat.

III. DEFINITION OF THE EXT FUNCTOR

Suppose that M is an arbitrary R-module and that the following is a projective resolution of M: $\ldots\to C_2 \to C_1 \to C_0 \to M \to 0$. We consider the nth right derived functor of Hom(_,N). This is denoted by $\mathrm{Ext}^n(M,N)$. If $0 \to M' \to M \to M'' \to 0$ is exact, we have the exact sequence, $0 \to \mathrm{Hom}(M'',N)\to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M',N) \to \mathrm{Ext}^1(M'',N) \to\mathrm{Ext}^1(M,N) \to\mathrm{Ext}^1(M',N) \to\ldots$.

From this, we obtain the following set of equivalent statements:

(i) M is projective.
(ii) $\mathrm{Ext}^n(M,N)=0$ for all N and all n>=1.
(iii) $\mathrm{Ext}^1(M,N)=0$ for all N.

We first determine as to when there exist projective modules that are direct summands of R. In this respect, we obtain the result that follows.

> *Result*: There exist projective modules that are direct summands of R if and only if R contains idempotents other than 0 and 1.

Suppose that R can be written as a direct sum I⊕J, where I and J are submodules (and hence ideals of R). Then 1 can be written as a sum i+(1-i), where i is in I and 1-i is in J. Consider the element i(1-i). This lies in both I and J and hence must be 0. Thus i is an idempotent in R. If i is 0 or 1, one of the submodules I and J will contain 1 and hence will be equal to R. Thus, R contains an idempotent not equal to 0 or 1.

Conversely, if we assume that R contains an idempotent $j \neq 0,1$ we always have a nontrivial projective module which is a direct summand of R. Consider the ideals (j) and (1-j). None of these modules is zero and hence it is enough to show that their intersection is 0. Say that jx = (1-j)y. Then y = j(x+y). Then jx = (1-j)j(x+y). But j(1-j) = 0 and hence jx = 0. Thus, the intersection is (0).

[1] We note that the RHS is symmetric in E and F, while the LHS is apparently not so. We have Hom(E,Hom(F,G)) ≅ Hom(F,Hom(E,G)).

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & 0 & & Tor(N,M) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & M'\otimes N' & \rightarrow & F_M\otimes N' & \rightarrow & M\otimes N' & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M'\otimes F_N & \rightarrow & F_M\otimes F_N & \rightarrow & M\otimes F_N & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & M'\otimes N & \rightarrow & F_M\otimes N & \rightarrow & M\otimes N & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Figure 1. A ‘tensor product’ of the sequences in (1).

IV. THE TOR FUNCTOR AND FLATNESS

a. Symmetry of the Tor Functor

The Tor functor directly measures the degree to which a module is flat. As the tor is part of a family of functors, it lends itself to defining flatness in a more general setting. First we show that the tor functor is symmetric: Tor(M,N) ≅ Tor(N,M).

Represent M and N as quotients of flat modules,

$$\begin{array}{l} 0 \rightarrow M' \rightarrow F_M \rightarrow M \rightarrow 0 \\ 0 \rightarrow N' \rightarrow F_N \rightarrow N \rightarrow 0 \end{array} \qquad (1)$$

We now ‘tensor the sequences with each other’—as shown in Figure 1, above. The snake lemma yields the exact sequence, $0 \rightarrow Tor(N,M) \rightarrow M'\otimes N \rightarrow F_M\otimes N$. But we already have,

$$0 \rightarrow Tor(M,N) \rightarrow M'\otimes N \rightarrow F_M \otimes N,$$

and hence Tor(M,N) ≅ Tor(N,M). An R-module M is flat if and only if Tor(N,M) = 0 for all R-modules N. If we have a projective resolution of N, say,

$$\ldots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow N \rightarrow 0,$$

then Tor is the ‘1st’ derived functor of the complex derived by tensoring the above resolution by M. Similarly Tor_n is the nth derived functor of the complex. Derived functors (of additive functors) are independent of the resolution chosen and hence we have the following set of equivalent statements:

(i) M is a flat module.
(ii) $Tor_i(N,M)=0$ for all R-modules N and all i>0.
(iii) $Tor_1(N,M)=0$ for all R-modules N.

b. Isomorphisms

Now that we have defined both Ext and Tor, we obtain the following isomorphisms:

(a) For any projective module P: Hom(P,Ext(F,G))≈Ext(P⊗F,G) for all F and G.
(b) For any injective module Q: Hom(Tor(E,F),Q)≈Ext(F,Hom(E,Q)) for all F and finitely generated E.

To prove (a) we start with the exact sequence: $0 \rightarrow G \rightarrow In \rightarrow I \rightarrow 0$, where In is an injective module. (*n.b.,* Any module can be embedded in an injective module.) We obtain: 0 → Hom(F,G) → Hom(F,In) → Hom(F,I) → Ext(F,G) → 0. Since P is projective, we have: 0→Hom(P,Hom(F,G)) →Hom(P, Hom(F,In)) → Hom(P, Hom(F,I)) →Hom(P,Ext(F,G)) →0. But we can directly obtain: 0 → Hom(P⊗F,G) → Hom(P⊗F,In) →Hom(P⊗F,I) → Ext(P⊗F,G) → 0. Since the first three terms of the two exact sequences are isomorphic, we have the result of part (a).

To prove part (b), we start with the exact sequence: 0 → I → Fr → E → 0, where Fr is a free module. (*i.e.* We write E as the quotient of a free module.) We tensor the sequence with F to obtain: 0 → Tor(E,F) → I ⊗F → Fr⊗F → E⊗F → 0.

Applying the functor Hom(_,Q) we obtain 0 → Hom(E⊗F,Q) → Hom(Fr⊗F,Q) →Hom(I⊗F,Q)→Hom(Tor(E,F),Q) → 0. But, we also have: 0 → Hom(E,Q) → Hom(Fr,Q) → Hom(I,Q) →0. And we get: 0→ Hom(F,Hom(E,Q)) → Hom(F,Hom(Fr,Q)) → Hom(F, Hom(I,Q)) → Ext(F,Hom(E,Q)) → 0. Hom(Fr,Q) is isomorphic to a direct sum of finitely many copies of Q and hence is injective (since E is finitely generated, Fr may be taken to be of finite basis). From the isomorphism of the first three terms in the two exact sequences, we obtain part (b).

c. Result of the Long Exact Sequence

First, we note that the long exact sequence of homology gives us the following: If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence, then the following is exact: $\ldots\to Tor_2(M',N) \to Tor_2(M,N) \to Tor_2(M'',N) \to Tor_1(M',N)\to Tor_1(M,N)\to Tor_1(M'',N)\to M'\otimes N \to M\otimes N\to M''\otimes N\to 0$. If N is an R-module such that $Tor_j(M,N)=0$ for all modules M, it is easy to see that $Tor_k(M,N)=0$ for all modules M and all k>j. This follows easily from the above long exact sequence. Any module M can be written as the image of a free module F and if we tensor that exact sequence by N to obtain its long exact sequence, we have the result.

d. Follow-on Definitions and Observations

We therefore define the following:

- A module M is k-flat if $Tor_k(N,M)=0$ for all R-modules N. (And we define k-projective and k-injective likewise with Ext^k)

Analogously, we have the following set of equivalent statements:

(i) M is k-flat.
(ii) $Tor_k(N,M)=0$ for all modules N.
(iii) $Tor_j(N,M)=0$ for all N and all j>=k.

Thus, if $0\to N'\to N\to N''\to 0$ is an exact sequence and M is a 2-flat module, it is carried to the exact sequence:

$0\to Tor(N',M)\to Tor(N,M)\to Tor(N'',M)\to N'\otimes M \to N\otimes M\to N''\otimes M\to 0$

For 3-flat modules the sequence will have the Tor_2 terms; for 4-flat modules the sequence will have Tor_3 terms and so on.

e. Results for k-flat Modules

Now we have an *if and only if* condition for k-flat modules.

Result: " A module M is k-flat if and only if $Tor_{k-1}(I,M)=0$ for all submodules I of free modules."

In order to demonstrate this, we take an arbitrary module N and write it as the quotient of a flat (free) module F: $0\to I\to F\to N\to 0$. We tensor the sequence by M and write out the long exact sequence:
$\to\ldots\to 0\to Tor_k(N,M)\to Tor_{k-1}(I,M)\to 0\to\ldots\to Tor_1(N,M)\to I\otimes M\to F\otimes M\to N\otimes M\to 0$.

Clearly, $Tor_k(N,M)=0$ iff $Tor_{k-1}(I,M) = 0$. Hence, M is k-flat if and only if $Tor_{k-1}(I,M) = 0$ for all submodules I of free modules.

We can also take another look at the result: If Hom(F,Q) is injective for each injective module Q, then F is flat. We can extend this to the following:

Result: If F is finitely generated F is 2-flat if and only if Hom(F,Q) is 2-injective for each injective module Q.

If F is 2-flat, it must be the quotient of a free module Fr with respect to a flat module I:$0 \to I\to Fr\to F\to 0$. If Q is injective, we obtain the sequence:
$0\to Hom(F,Q)\to Hom(Fr,Q) \to Hom(I,Q) \to 0$
From this we obtain: $Ext(E,Hom(I,Q))\to Ext^2(E,Hom(F,Q)) \to Ext^2(E, Hom(Fr,Q)) \to Ext^2(E, Hom(I,Q))$.
The first and last terms are zero because Hom(I,Q) is injective (since I is flat). Also Hom(Fr,Q) is isomorphic to a direct sum of finitely many copies of Q and hence is injective. Thus $Ext^2(E,Hom(F,Q)=0$ for any E. Thus Hom(F,Q) is 2-injective.

Conversely, suppose that Hom(F,Q) is 2-injective for any injective module Q. Let I and Fr be defined as in the previous part. We can obtain the exact sequence:

$0\to Hom(F,Q)\to Hom(Fr,Q)\to Hom(I,Q)\to 0$.

Applying the functor Hom(E,_), we get: $Ext(E,Hom(Fr,Q)) \to Ext(E,Hom(I,Q)) \to Ext^2(E,Hom(F,Q)) \to Ext^2(E,Hom(Fr,Q))$. The first and last terms are zero. Since Hom(F,Q) is 2-injective, the third term is also zero. Thus Hom(I,Q) is injective. Hence I is flat. Thus, F is 2-flat.

If a module M is 2-flat, we want to find conditions under which it becomes flat (or 1-flat). Accordingly, we have the following result:

Result: The following statements are equivalent:
(ii) M is flat.
(iii) M is 2-flat and Tor(Q,M)=0 for all injective modules Q.
(i)$\Rightarrow$(ii) is obvious.

To prove the second part, we use the fact that every module can be embedded into an injective module. Let M be 2-flat and N be an arbitrary R-module. Then there exists an injective module Q such that N can be embedded in Q. Thus, there is an exact sequence $0\to N\to Q\to Q/N\to 0$. Since M is 2-flat, we have the sequence:

$0\to Tor_1(N,M)\to Tor_1(Q,M)\to Tor_1(Q/N,M)\to N\otimes M\to\ldots$

If we allow Tor(Q,M) = 0 for all injective modules Q, we have Tor(N,M) = 0 for all modules N. Hence, M is flat.

Each of the properties (flatness, 2-flatness and so on is a local property:

Result: The following statements are equivalent:
(i) M is k-flat.
(ii) M_p is k-flat for each prime ideal p.
(iii) M_m is k-flat for each maximal ideal m.

Let $\ldots\to C_2\to C_1\to C_0\to N\to 0$ be a projective resolution of an R-mdoule N. We form the complex M⊗C. Now, $Tor_k(N,M)$ is the nth homology module of this complex and its localizations are the homology modules of the complex after localization. Since Q = 0 $\Leftrightarrow$ each $Q_p = 0$ $\Leftrightarrow$ each $Q_m = 0$ for any module Q, and we have our result.

V. THEOREM

Theorem: Let M be an R-mdoule. If I is an ideal of R, then the map $I\otimes M\to M$ is an injection if and only if Tor(R/I,M)=0. The module M is flat if and only if this is so for every ideal I.

Proof: Consider the exact sequence: $0\to I\to R\to R/I\to 0$. We obtain the exact sequence: $0\to Tor(R/I,M)\to I\otimes M\to R\otimes M$.
Since R⊗M=M, Tor(R/I,M) is the kernel of the map from I⊗M to M. Thus the map is an injection if and only if Tor(R/I,M)=0.

Now, assume that Tor(R/I,M)=0 for all ideals I. Suppose that $P\to Q$ is an injection of R-modules $M\otimes P\to M\otimes Q$ is not an injection. Then there exists a non-zero element m⊗p that goes to zero. If we restrict the map to the module generated by the finitely many elements required to take m⊗p to 0, we obtain a finitely generated module for which this map is not an injection. But every finitely generated module can be decomposed into a finite chain of submodules, the successive quotients of which are cyclic modules and hence isomorphic to some R/I. Thus, Tor(R/I,M)=0 for each I implies that M is flat.

VI. FLATNESS OF PROJECTIVE MODULES

The above theorem shows that in order to check whether a module is flat, we need to consider the tensor products I⊗M, where I is an ideal. Suppose that x is a non-zero element of M and a is an element such that ax=0. Clearly, a is a nonunit and hence there exists a maximal ideal m containing x. Since the map from m⊗M to M must be injective, we must have a⊗x=0 in m⊗M. In general, we can say that if I is an ideal and $\sum a_i x_i=0$ with $a_i\in I$ we must have $\sum a_i\otimes x_i=0$. In this respect, we state the following criterion (without proof) of when an element of the tensor product of two modules M and N is zero.

Criterion: Let N be generated by a set of elements $\{n_i\}$. Every element of M⊗N may be written as a finite sum $\sum m_i\otimes n_i$, where the m_i lie in M. Such an expression is 0 iff there exist elements m_j' of M and elements a_{ij} of R such that $\sum a_{ij}m_j'=m_i$ for each i (sum is taken over j) and $\sum a_{ij}n_i=0$ for each j (sum is taken over i).

Projective modules are always flat. This follows from the fact that a direct sum (finite or infinite) of R-modules is flat if and only if each of its direct summands is finite. This result can be easily generalized:

Result: A direct sum of R-modules is k-flat if and only each of its direct summands is k-flat.

We will actually prove that $Tor_k(P,M\oplus N)=0$ iff $Tor_k(P,M)=0$ and $Tor_k(P,N)=0$. This is true for k=1. We assume it to be true up to k-1. Express P as a quotient of a free module: $0\to I\to F\to P\to 0$ and consider the exact sequence

$0\to Tor_k(P,M\oplus N)\to Tor_{k-1}(I,M\oplus N)\to 0$.

If $Tor_k(P,M\oplus N)=0$, we have $Tor_{k-1}(I,M\oplus N)=0$ which means that $Tor_{k-1}(I,M)=0$ and so also for N. From the sequence obtained by substituting M for M⊕N in the above sequence, we have $Tor_k(P,M)=0$. The converse is proved similarly.

We note that properties of flat modules tend to carry over to k-flat modules in an analogous manner. Let us generalize the statement "Projective modules are flat".

Result: k-projective modules are k-flat.

Let M be k-projective, i.e. $Ext^k(M,N)=0$ for all N. Let k>1. We write M as the quotient of a free module F: $0 \to I \to F \to M \to 0$. We derive the exact sequence:
$0\to Ext^{k-1}(I,N) \to Ext^k(M,N) \to 0$ for any module N. Thus $Ext^{k-1}(I,N)=0$ for all N and hence I is k-1 projective. We know that projective modules are flat and hence we may assume the statement to be true up to k-1. Thus I is k-1 flat. Now, we also have the exact sequence: $0\to Tor_k(M,N) \to Tor_{k-1}(I,N) \to 0$. Since I is k-1 flat, $Tor_{k-1}(I,N)=0$ for all N and hence $Tor_k(M,N)=0$ for all N. Thus, M is k-flat.

VII. FLATNESS IN NOETHERIAN MODULES AND LOCAL RINGS

Let us assume that R, in addition to being a commutative ring with identity, satisfies the noetherian condition.

If M is a finitely generated module over R, M is flat if and only if Tor(R/I,M)=0 for each ideal of R.

Result: The following statements are equivalent:

(i) Every finitely generated R-module is flat.

(ii) Each of the quotient modules R/p (p a prime ideal) is flat.

(iii) Every R-module is flat, *i.e.*, R is absolutely flat.

To prove that (ii)⇒(i), we use the fact that every finitely generated R-module can be filtered as: $M=M_0\supset M_1\supset\ldots\supset M_k=0$ where successive quotients are isomorphic to R/p for some prime ideal p. Thus, M_{k-1} is flat. Since M_{k-2}/M_{k-1} is flat, we have proved that M_{k-2} is flat. Proceeding thus, we can show that M is flat.

To prove that (i)⇒(iii), we use the fact that an R-module M is flat iff Tor(R/I,M)=0 for each I. Since R/I is finitely generated, R/I is flat. Thus, Tor(R/I,M)=0 for each M and hence M is flat.

Result: The following statements are equivalent:

(i) M is flat.

(ii) M is 2-flat and Tor(R/q,M)=0 for each primary ideal q of R.

Let M be a non-zero cyclic module over a local ring (R,M). Then M is isomorphic to some R/I. Let us assume that I is non-zero. Suppose that M is flat. Take an element a in I such that ax=0, where x is the generator of the cyclic module M. Since M is flat, there exist elements $b_1, b_2\ldots bk$ in R and elements $y_1,y_2\ldots y_k$ in M such that $b_ja=0$ for each $0<j<=k$ and $x=\sum b_jy_j$. Let $y_j=r_jx$ and $c_j=b_jr_j$. Then, each $c_ja=0$ for each j and x is annihilated by $t=1-\sum c_j$. Note that each of the c's lies in Ann(a). We assume that Ann(a) is contained in m. Then t is not in m and hence t is a unit. Since tx=0, we have x=0. This is a contradiction. Hence Ann(a)=R and a=0. Thus, if M is same as R/I, I cannot contain a non-zero element. Thus *the only cyclic and flat modules over a local ring are R and 0*.

The above result can be restated for a general ring: If M is a non-zero, cyclic and flat module over a ring R and a annihilates M, Ann(a) is not contained inside the Jacobson radical of R. In other words, if R/I is flat, Ann(i) is not contained inside the Jacobson radical for any i in I.

Suppose that R/I is flat and I is neither 0 nor R. Hence the localization $(R/I)_m$ where m is any maximal ideal, is flat. Now R_m/I_m is a cyclic module over the local ring R_m. Hence, it must be either 0 or R_m. Thus, we have $I_m=R_m$ or $I_m=0$. If m does not contain I, we have $I_m=R_m$. Hence, we have the result:

Result: R/I is flat over R if and only if $I_m=R_m$ or $I_m=0$ for each maximal ideal m of R containing I.

The "only if" part has been proved above. The "if" part is obvious since M is flat ⇔ each M_m is flat.

We may extend this procedure to flat modules having a <u>minimal</u> generating set consisting of two elements, say m_1 and m_2. Suppose that $n_1m_1+n_2m_2=0$. Then there exist elements a_{ij} in R, i=1,2, j=1,2…k such that

$$a_{1j}n_1+a_{2j}n_2=0 \text{ for each j.}$$

and

$$(\sum a_{1j}r_j -1)m_1+(\sum a_{1j}s_j)m_2=0$$
$$(\sum a_{2j}r_j)m_1+(\sum a_{2j}s_j -1)m_2=0.$$

Now suppose that R is a local ring. Consider the ideal I generated by elements a where for each a, there exists b such that

$an_1+bn_2=0$. If this ideal is contained in the maximal ideal of R, the coefficient of m_1 in the 1st equation becomes a unit and hence $\{m_1,m_2\}$ is not a minimal generating set. Thus, we can say that I=R. The same holds for the second "co-ordinate". This process can be extended to higher dimensions. Thus, we can develop a necessary condition for flatness of a finitely generated module over a local ring. This may be extended to a general R.

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