A deterministic model for (*n = 2*) competitive products in a market system

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ABSTRACT

We proposed a new deterministic model for the dynamics of two competitive products in a given market system. The model was analyzed qualitatively to determine the stability of its equilibrium under the influence of factors such as advertisement, personal interaction, immigration and emigration. Numerical verification of the analytical results is performed and presented graphically.

I. INTRODUCTION

The diffusion of competitive products major forecast in marketing research as deterministic models of new competitive products have been studies in [1-9]. The main aim objective for this work is to construct mathematical diffusion model for two competitive products in any given market. The major impetus behind this research has been the perceived high failure rate of new competitive products and the consequent need to improve the marketing decision concerning with the launch and adoption of such products. The key factors contributing to competitive products adoption in any given market is advertisement and personal interaction. Lack of intensifying factors such as advertisement of the product can lead to the product failing to sustain itself in a competitive market.

A major problem of interest in any given market is that of forecasting on how people react to introduction of competitive product. Generally, apart from advertisement and personal interaction, immigration and emigration of population also contribute positively or otherwise to the spread or adoption of competitive products in a given market system.

a. Local Behavior of Autonomous System

i. Routh-Hurwitz Criterion

The roots of the characteristic polynomial

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-1}\lambda^{n-2} + \ldots + a_0 \quad (1)$$

and

$$m_1 = \left(a_{n-1}\right), \qquad m_2 = \begin{pmatrix} a_{n-1} & a_{n-3} \\ 1 & a_{n-2} \end{pmatrix}$$

$$m_k = \begin{pmatrix} a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & . & . & . & a_{-n+1} \\ 1 & a_{n-2} & a_{n-4} & a_{n-6} & . & . & . & a_{-n+2} \\ 0 & a_{n-1} & a_{n-3} & . & . & . & . & a_{-n+3} \\ 0 & 1 & a_{n-2} & . & . & . & . & a_{-n+4} \\ 0 & 0 & a_{n-1} & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & 0 & . & . & . & . & a_{-1} \\ 0 & 0 & 0 & 0 & . & . & . & a_0 \end{pmatrix},$$

ii. Dulac's Criterion

In a simply connected region there exists smooth function

$$\rho(x_1, x_2),$$

such that the expression

$$div(\rho f) = \frac{\partial(\rho f_1)}{\partial x_1} + \frac{\partial(\rho f_2)}{\partial x_2}, \qquad (2)$$

is of one sign, then there are no periodic orbits in . Such a function is called a Dulac function.

II. FORMULATION OF THE MODEL

In the proposed mathematical model it is assumed that the any given market system population is divided into three categories, namely the non-users of the product (N), adopters of first product (A_1), adopters of the second product (A_2). In addition we assume that each user adopts one product at a time. The immigration population is assumed to be non-users while the emigration population may consist of users and non-users.

$$\frac{dA_1}{dt} = (\gamma_1 + \lambda_1 A_1)N - (\delta + \nu_1)A_1, \qquad (3a)$$

$$\frac{dA_2}{dt} = (\gamma_2 + \lambda_2 A_2)N - (\delta + \nu_2)A_2, \qquad (3b)$$

$$\frac{dN}{dt} = \beta - \sum_{i=1}^{2}(\gamma_i + \lambda_i A_i)N + \sum_{i=1}^{2} \nu_i A_i - \delta N, \qquad (3c)$$

Where A_1 and A_2 are adopters of the first product and second product, respectively, λ is contact rate between users and non-users, γ is advertising intensity of the products, ν is aborting rate of the product, β is the immigration rate of population, and δ is the emigration rate of population.

III. QUALITATIVE RESULTS

Revealed qualitative results were put in form of theorems.

Theorem 1. The total population in a given market system is equal to the ratio of immigration to that of emigration after a long period of time.

Proof.

$$\frac{dA_1}{dt} = (\gamma_1 + \lambda_1 A_1)N - (\delta + \nu_1)A_1, \qquad (4)$$

$$\frac{dA_2}{dt} = (\gamma_2 + \lambda_2 A_2)N - (\delta + \nu_2)A_2, \qquad (5)$$

$$\frac{dN}{dt} = \beta - \delta N - \lambda_1 A_1 N - \lambda_2 A_2 N - \gamma_1 N - \gamma_2 N + \nu_1 A_1 + \nu_2 A_2, \qquad (6)$$

Adding (4), (5) and (6) gives

$$\frac{dP}{dt} + \delta P = \beta, \text{ where } P = N + A_1 + A_2$$

Which yields,

$$P(t) = \frac{\beta}{\delta} + De^{-\delta t},$$

As $t \to \infty$, then $P(t) = \dfrac{\beta}{\delta}$. ■

Theorem 2. There exists a positive equilibrium (A_1^*, A_2^*) provided all the parameters are positive.

Re-write the model as follows

$$\frac{dA_1}{dt} = (\gamma_1 + \lambda_1 A_1)(C - A_1 - A_2) - \alpha_1 A_1, \qquad (7)$$

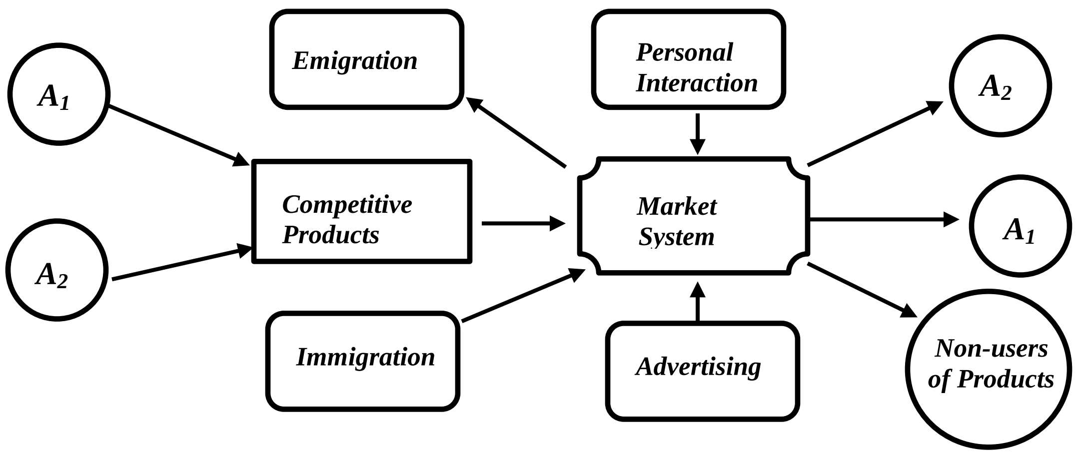

Figure 1. Schematic model for two competitive products.

$$\frac{dA_2}{dt} = (\gamma_2 + \lambda_2 A_2)(C - A_1 - A_2) - \alpha_2 A_2, \tag{8}$$

where $C = \dfrac{\beta}{\delta}$ and $\alpha_i = (\delta + \upsilon_i)$.

Proof: Equating (7) and (8) to zero, we obtain

$$\frac{\alpha_1 A_1}{(\gamma_1 + \lambda_1 A_1)} = \frac{\alpha_2 A_2}{(\gamma_2 + \lambda_2 A_2)} = C - A_1 - A_2, \tag{9}$$

Writing A_1 in terms of A_2 then, we obtain

$$A_1 = \frac{\alpha_2 \gamma_1 A_2}{\alpha_1 \gamma_2 + (\alpha_1 \lambda_2 - \alpha_2 \lambda_2) A_2}.$$

Let

$$q_1(A_2) = \frac{\alpha_2 \gamma_1 A_2}{\alpha_1 \gamma_2 + (\alpha_1 \lambda_2 - \alpha_2 \lambda_2) A_2},$$

and

$$q_2(A_2) = \frac{\alpha_2 A_2}{(\gamma_2 + \lambda_2 A_2)},$$

evaluating $q_1'(A_2)$ *and* $q_2'(A_2)$ yields

$$q_1'(A_2) = \frac{\alpha_1 \alpha_2 \gamma_2^2}{[\alpha_1 \gamma_2 + (\alpha_1 \lambda_2 - \alpha_2 \lambda_1) A_2]^2} > 0,$$

$$q_2'(A_2) = \frac{\alpha_2 \gamma_2}{(\gamma_2 + \lambda_2 A_2)^2} > 0.$$

Since $q_1'(A_2)$ and $q_2'(A_2)$ are greater than zero, hence $q_1(A_2)$ and $q_2(A_2)$ are continuous and strictly monotonically increasing functions of A_2 on $\left[0, \dfrac{\beta}{\delta}\right]$.

Note: $q_1(A_2) = A_1^*$ since $q_1'(A_2) > 0$ then people will still be adopting A_1^* as $(t \to \infty)$.

Similarly: From (9) we write A_2 in terms of A_1

$$A_2 = \frac{\alpha_2\gamma_1 A_1}{\alpha_2\gamma_1 + (\alpha_2\lambda_1 - \alpha_1\lambda_2)A_1}.$$

Let

$$q_1(A_1) = \frac{\alpha_2\gamma_1 A_1}{\alpha_2\gamma_1 + (\alpha_2\lambda_1 - \alpha_1\lambda_2)A_1},$$

and

$$q_2(A_1) = \frac{\alpha_1 A_1}{(\gamma_1 + \lambda_1 A_1)},$$

Evaluating $q_1^{'}(A_1)$ and $q_2^{'}(A_1)$ yields

$$q_1^{'}(A_1) = \frac{\alpha_1\alpha_2\gamma_1\gamma_2}{[\alpha_2\gamma_1 + (\alpha_2\lambda_1 - \alpha_1\lambda_2)A_2]^2} > 0,$$

$$q_1^{'}(A_1) = \frac{\alpha_1\gamma_1}{(\gamma_1 + \lambda_1 A_1)^2} > 0.$$

Note: $q_1(A_1) = A_2^*$ since $q_1^{'}(A_1) > 0$ then people will still be adopting A_2^* as $(t \to \infty)$. ■

Theorem 3. (A_1^*, A_2^*) is locally asymptotically stable provided all parameters are positive.

Let

$$f_1 = (\gamma_1 + \lambda_1 A_1)(C - A_1 - A_2) - \alpha_1 A_1, \tag{10}$$

$$f_2 = (\gamma_2 + \lambda_2 A_2)(C - A_1 - A_2) - \alpha_2 A_2, \tag{11}$$

Proof. The Jacobian matrix here is

$$J_E = \begin{pmatrix} -\beta_1 & -b_1 \\ -b_2 & -\beta_2 \end{pmatrix}$$

where $b_i = \gamma_i + \lambda_i A_i^*$, $\beta_i = b_i + d_i$, and

$$d_i = \frac{\alpha_i\gamma_i}{\gamma_i + \lambda_i A_i^*}.$$

The characteristic polynomial

$$p(r) = r^2 + a_1 r + a_0,$$

where

$$a_1 = (\beta_1 + \beta_2) > 0,$$

$$a_0 = \beta_1\beta_2 - b_1 b_2,$$

$$= (b_1 + d_1)(b_2 + d_2) - b_1 b_2,$$

$$b_1 d_2 + d_1 b_2 + d_1 d_2 > 0.$$

Since this is a 2 x 2 matrix and all the coefficients of the characteristics polynomial are strictly positive, then all the eigenvalues must have strictly negative real part, this can also be confirmed using the Routh-Hurwitz theorem even higher dimension. Hence the system is locally asymptotically stable.

Theorem 4. (A_1^*, A_2^*) is globally asymptotically stable provided all parameters are positive.

Dulac's Criterion: If in a simply connected region D there exists a smooth function $M(A_1, A_2)$, such that the expression

$$div\,(Mf) = \frac{\partial(Mf_1)}{\partial A_1} + \frac{\partial(Mf_2)}{\partial A_2},$$

is of one sign, then there are no periodic orbits in D.

Proof. Using Dulac's Criterion and according to the theory about global stability of a competitive dynamical system [10], $E\left(A_1^*, A_2^*\right)$ is globally asymptotically stable if there are no periodic solutions, including limit cycles, hormoclinic orbits and oriented phase polygons in region $D = \{(A_1, A_2); 0 \le A_1 + A_2 \le C\}$.

Let

$$f_1 = (\gamma_1 + \lambda_1 A_1)(C - A_1 - A_2) - \alpha_1 A_1, \tag{12}$$

$$f_2 = (\gamma_2 + \lambda_2 A_2)(C - A_1 - A_2) - \alpha_2 A_2, \tag{13}$$

and

$$M = \frac{1}{(\gamma_1 + \lambda_1 A_1)(\gamma_2 + \lambda_2 A_2)},$$

then

$$\frac{\partial(f_1 M)}{\partial A_1} + \frac{\partial(f_2 M)}{\partial A_2},$$

$$-\left[\frac{\left(2\lambda_1 A_1 \gamma_1 + \gamma_1^2 + \alpha_1 \gamma_1\right)}{(\gamma_1 + \lambda_1 A_1)^2 (\gamma_2 + \lambda_2 A_2)} + \frac{\left(2\lambda_2 A_2 \gamma_2 + \lambda_2^2 A_2^2 + \gamma_2^2 + \alpha_2 \gamma_2\right)}{(\gamma_1 + \lambda_1 A_1)(\gamma_2 + \lambda_2 A_2)^2}\right] < 0.$$

Hence the Dulac criterion is satisfied and the system is globally asymptotically stable. ■

Theorem 5. Let $\gamma_1 = 0$, $\gamma_2 > 0$, then there exists two possible positive equilibria $E_0\left(0, A_2^*\right)$ and $E_1\left(A_1^*, A_2^*\right)$, such that

1. $E_0\left(0, A_2^*\right)$ is locally asymptotically stable provided $\lambda_2 \alpha_1 > \lambda_1 \alpha_2$.
2. $E_1\left(A_1^*, A_2^*\right)$ is locally asymptotically stable provided $\lambda_2 \alpha_1 < \lambda_1 \alpha_2$.

Proof. Consider

$$\frac{dA_1}{dt} = A_1\left[\lambda_1 (C - A_1 - A_2) - \alpha_1\right] = f_1(A_1, A_2),$$

$$\frac{dA_2}{dt} = (\gamma_2 + \lambda_2 A_2)\left[(C - A_1 - A_2) - \alpha_2 A_2\right] = f_2(A_1, A_2),$$

At equilibrium, $f_1 = 0$

==> $A_1 = 0$ or $C - A_1 - A_2 = \dfrac{\alpha_1}{\lambda_1}$,

$f_2 = 0$,

==> $C - A_1 - A_2 = \dfrac{\alpha_2 A_2}{\gamma_2 + \lambda_2 A_2}$.

Equilibrium point is $E_0\left(0, A_2^*\right)$

Where $C - A_2^* = \dfrac{\alpha_2 A_2^*}{\gamma_2 + \lambda_2 A_2^*}$.

We have also $E_1\left(A_1^*, A_2^*\right)$, where

$$A_2^* = \frac{\alpha_2 \gamma_2}{\alpha_2 \lambda_1 - \alpha_1 \lambda_2},$$

and $A_1^* = C - A_2^* - \dfrac{\alpha_1}{\lambda_1}$.

Test for local stability of E_0 and E_1

Consider $E_0\left(0, A_2^*\right)$

$$J_{E_0} = \begin{pmatrix} \lambda_1\left(C - A_2^*\right) - \alpha_1 & 0 \\ -\left(\gamma_2 + \lambda_2 A_2^*\right) & -\left(\gamma_2 + \lambda_2 A_2^*\right) + \lambda_2\left(C - A_2^*\right) - \alpha_2 \end{pmatrix}$$

The eigenvalues are

$$r_1 = \lambda_1\left(C - A_2^*\right) - \alpha_1,$$

$$= \frac{\lambda_2 \alpha_2 A_2^*}{\gamma_2 + \lambda_2 A_2^*} - \alpha_1$$

$$= \frac{(\lambda_1 \alpha_2 - \lambda_2 \alpha_1) A_2^* - \alpha_1 \gamma_2}{\gamma_2 + \lambda_2 A_2^*} < 0,$$

Provided $\lambda_1 \alpha_2 < \lambda_2 \alpha_{1,}$

$$r_2 = -\alpha_2 + \lambda_2 \left(C - A_2^* \right) - \left(\gamma_2 + \lambda_2 A_2^* \right) < 0,$$

Hence $E_0 \left(0, A_2^* \right)$ is locally asymptotically stable provided $\lambda_1 \alpha_2 < \lambda_2 \alpha_{1,}$

Consider $E_1 \left(A_1^*, A_2^* \right)$, where

$$A_2^* = \frac{\alpha_2 \gamma_1}{\alpha_2 \lambda_1 - \alpha_1 \lambda_2},$$ and

$$A_1^* = C - A_2^* - \frac{\alpha_1}{\lambda_1}.$$

The Jacobian matrix

$$J_{E_1} = \begin{pmatrix} \lambda_1 A_1^* & \lambda_1 A_1^* \\ -\left(\gamma_2 + \lambda_2 A_2^* \right) & \dfrac{(\lambda_2 \alpha_1 - \lambda_1 \gamma_2) - \lambda_1 \left(\gamma_2 + \lambda_2 A_2^* \right)}{\lambda_1} \end{pmatrix}.$$

Characteristics polynomial

$$P(r) = r^2 + a_1 r + a_0,$$

where

$$a_1 = \frac{\lambda^2 A_1^* + (\lambda_1 \alpha_2 - \lambda_2 \alpha_1) + \lambda_1 A_1^* \left(\gamma_2 + \lambda_2 A_2^* \right)}{\lambda_1} > 0,$$

and

$$a_0 = A_1^* (\lambda_1 \alpha_2 - \lambda_2 \alpha_1) > 0,$$

provided $\lambda_1 \alpha_2 > \lambda_2 \alpha_1$.

Since $a_1 > 0$ and $a_0 > 0$, then there exists eigenvalues with strictly negative real parts, Hence $E_1 \left(A_1^*, A_2^* \right)$ is locally asymptotically stable. ■

IV. NUMERICAL RESULTS

Numerical results on the model, was performed in order to validate the mathematical model and the results from the qualitative study. The numerical method used in the results shown in Figures 2-6 were obtained using the RK4 tool in MAPLE.

V. ANALYSIS

The analytical results agree with the numerical results. It is interesting to notice figure 2 a monotonic decrease in the non-users population of the products with time. However figures1, 3 and 4 a gradual increase in the adopter's populations is observed with time. This implies that some if the non-users are now becoming users as a results of positive influence if advertisement, personal interaction, immigration and emigration.

It is noteworthy as emulated in the qualitative results for both local and global stability analysis that absence if advertisement and personal interaction may lead to one of the product eventually decline in the adoption of products.

The other interesting observation is on figure advertisement the product will continue in the market provided there is positive personal interaction, immigration and emigration to a certain level in if advertisement is not intensify may lead to the product fading in a competitive market.

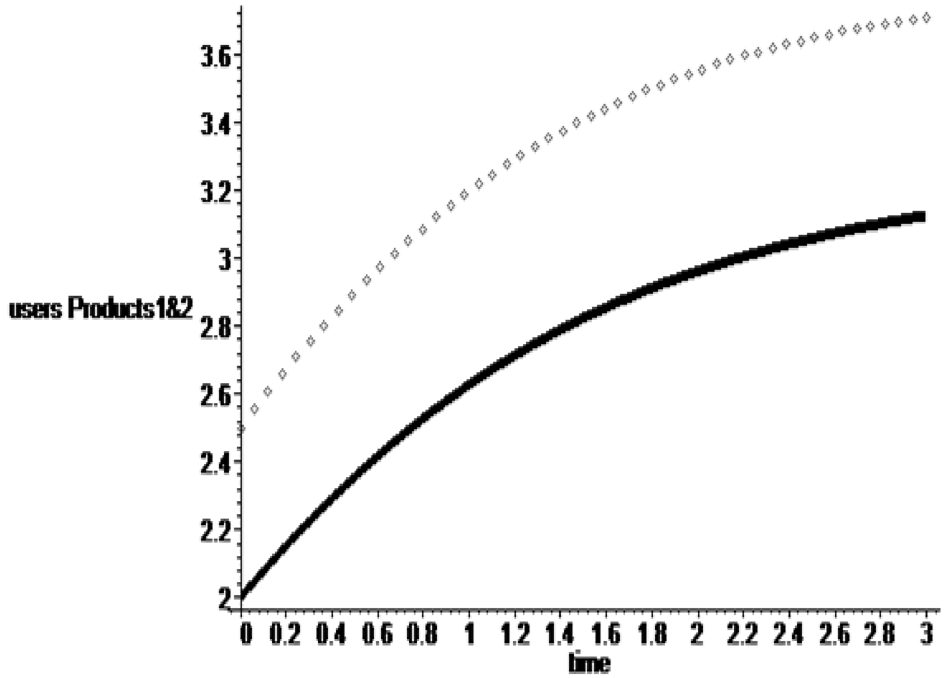

Figure 2. Variation of both Products users verses time with parameters ($\gamma_1 = \gamma_2 = 0.1$, $\lambda_1 = \lambda_2 = 0.1$, $\nu_1 = \upsilon_2 = 0.1$, $\delta = 0.1$ and $A1(0) = 2$, $A2(0) = 2.5$).

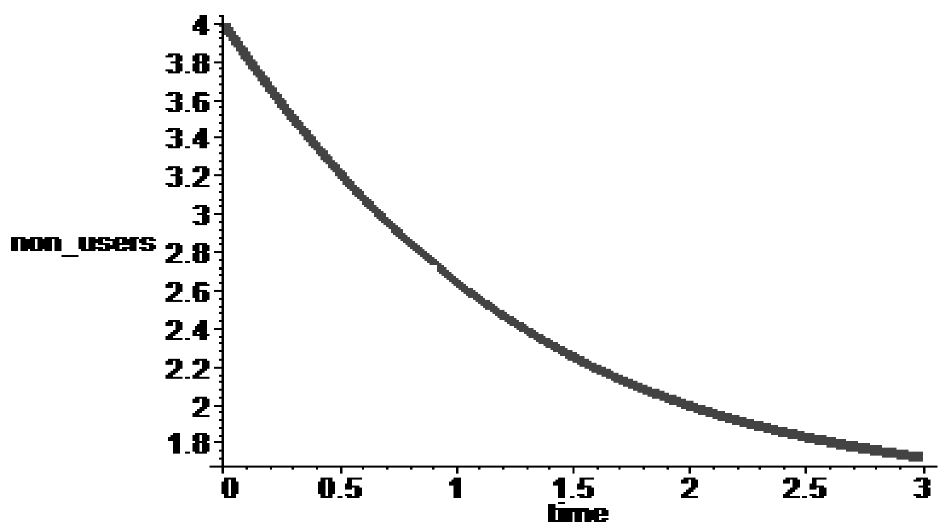

Figure 3. Variation of nonusers verses time with parameters ($\gamma_1 = \gamma_2 = 0.1$, $\lambda_1 = \lambda_2 = 0.1$, $\nu_1 = \upsilon_2 = 0.1$, $\delta = 0.1$, $\beta = 0.9$ and $N(0) = 4$).

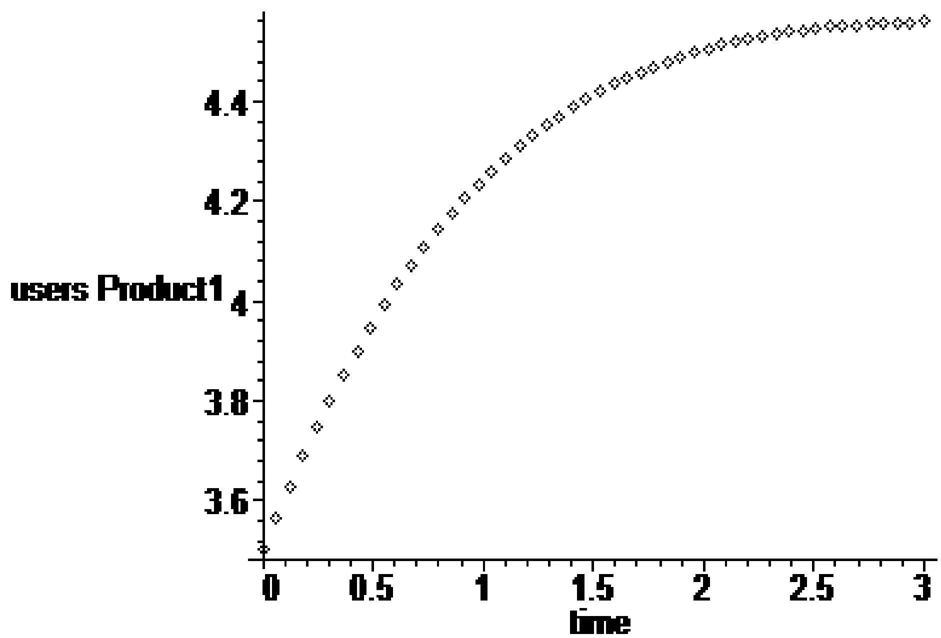

Figure 4. Variation of Products1 users verses time with parameters ($\gamma_1 = 0.1$, $\lambda_1 = 0.1$, $\nu_1 = 0.1$, $\delta = 0.1$ and $A1(0) = 2$).

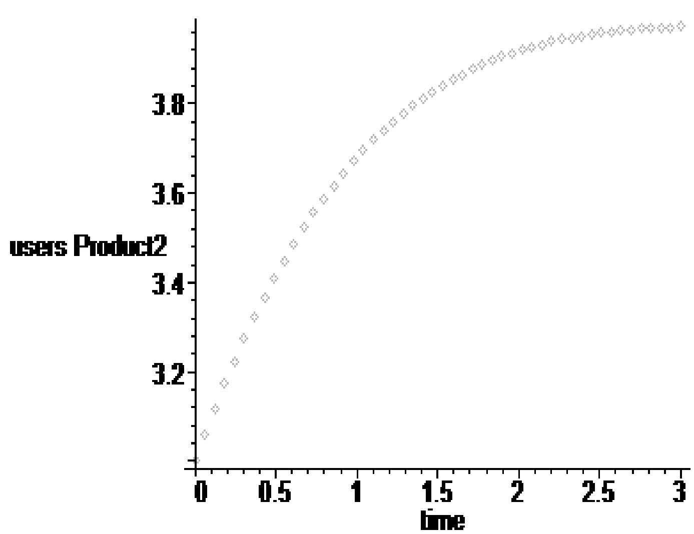

Figure 5. Variation of Products2 users verses time with parameters ($\gamma_2 = 0.1$, $\lambda_2 = 0.1$, $\nu_2 = 0.1$, $\delta = 0.1$ and $A1(0) = 3$).

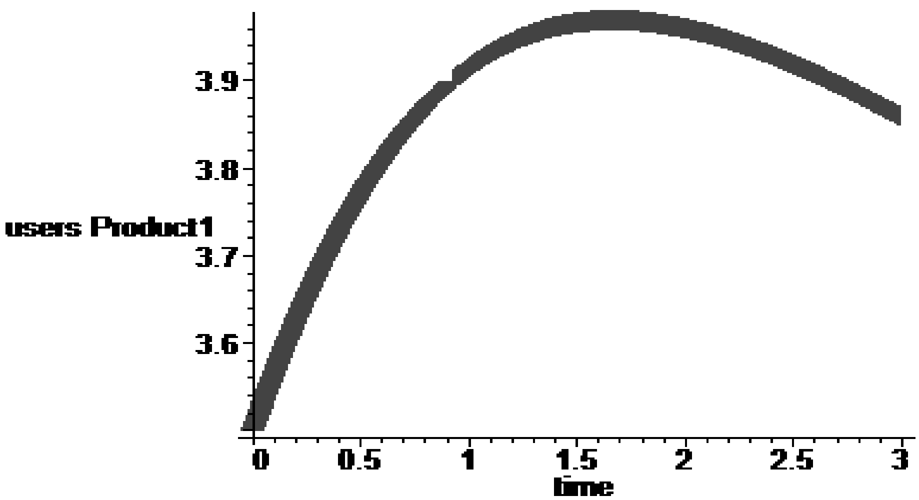

Figure 6. Variation of Products1 [No Advertisement] users verses time with parameters ($\gamma_1 = 0$, $\lambda_1 = 0.1$, $\nu_1 = 0.1$, $\delta = 0.1$ and $A1(0) = 2$).

VI. CONCLUSIONS

The qualitative results obtained using the local and global stability analysis concur with the numerical results obtained from Runge-Kutta4 from Maple software which indicate that in a competitive market system for the new competitive product to sustain itself there must be positive advertisement, personal interaction, immigration and emigration.

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